eureka

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Eureka 61

A Journal of The Archimedeans
Cambridge University Mathematical Society

Editors: Philipp Legner and Anja Komatar

October 2011

Editorial

Eureka Reinvented...

Before reading any part of this issue of Eureka, you will have noticed two big changes we have made: Eureka is now published in full colour, and printed on a larger paper size than usual. We felt that, with the internet being an increasingly large resource for mathematical articles of all kinds, it was necessary to offer something new and exciting to keep Eureka as successful as it has been in the past. We moved away from the classic LATEX-look, which is so common in the scientific community, to a modern, more engaging, and more entertaining design, while being conscious not to lose any of the mathematical clarity and rigour.

To make full use of the new design possibilities, many of this issue's articles are based around mathematical images: from fractal modelling in financial markets (page 14) to computer rendered pictures (page 38) and mathematical origami (page 20). The *Showroom* (page 46) uncovers the fundamental role pictures have in mathematics, including patterns, graphs, functions and fractals.

This issue includes a wide variety of mathematical articles, problems and puzzles, diagrams, movie and book reviews. Some are more entertaining, such as *Bayesian Bets* (page 10), some are more technical, such as *Impossible Integrals* (page 80), or more philosophical, such as *How to teach Physics to Mathematicians* (page 42). Whether pure or applied, mathematician or not, there will be something interesting for everyone. If you don't know where to start reading, skim through the pages and have a look at the *Number Dictionary* at the bottom.

If you have any comments, would like to write an article for the next issue, or be part of the next production team, please do not hesitate to contact us or any member of the Archimedeans committee.

I hope you will enjoy reading the 61st issue of Eureka!

Philipp Legner

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38 Painting with Pixels

Contents

The Archimedeans

Philipp Kleppmann, President 2011 – 2012

The highlight of this year's Archimedeans' calendar was the black-tie Triennial Dinner. Seventy-five members and guests attended the champagne reception and excellent meal in the Crowne Plaza Hotel to celebrate another stepping stone in the long life of the society.

We hosted several academic talks this year, starting with Prof. Siksek's account of Diophantine equations, their history, and methods of solving these deceptively simple-looking equations. Our main theme were the *Millennium Prize Problems*, with five engaging talks throughout the year. These problems where set by the Clay Mathematics Institute in 2000, and only one of them has since been resolved. The talk by Prof. Donaldson on the Poincaré Conjecture proved to be especially popular. Prof. Donaldson was on the Advisory Board that recommended Dr. Perelman as prize winner, and is an expert in the area.

At the end of term we organised the annual problems drive. The questions were set by last year's winners, and people from as far away as Warwick took part in this light-hearted competition. The questions can be seen on the following pages; points were also awarded for the funniest and most creative answers.

The first event in Easter Term was a relaxing punting trip immediately after the exams. A little later we joined in the general madness of May Week with the Science Societies' Garden Party. Novelties this year were live music and a cheese bar – both went down very well!

This has been a fantastic year, both for our members and for the committee, and we are looking forward to another interesting year. I would like to thank the committee for their dedication, and the members and subscribers of Eureka for their continued support.

The Committee 2011 – 2012

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Maximum number of colours needed to colour any map in the Euclidean plane.

THE AR

◀ Archimedeans Garden Party 2011

Talk by Prof. ▶ Béla Bollobás

CHI

▲ Party after Prof. Bollobás' Talk

▲ Archimedeans Garden Party 2011

▲ Post-Exams Punting 2011

MEDE

▲ Fundamental Theorem of Comedy

ANS

◀ Post-Exams Punting 2011

Introduction to Raptor Theory

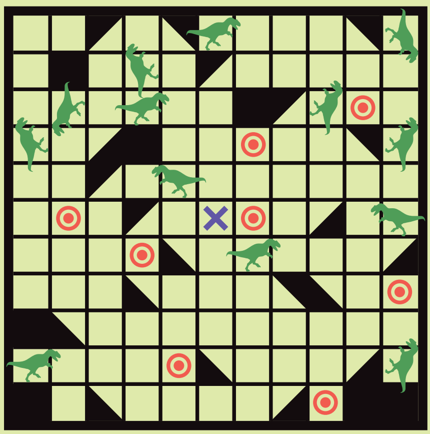
Archimedeans Problems Drive 2011
by Mary Fortune and Jonathan Lee

1 Attitude Adjuster

A velociraptor spots you 40 meters away and attacks, accelerating at 4 m/s^2 from a stand start, up to its top speed of 25 m/s. When it spots you, you begin to flee, quickly reaching your top speed of 6 m/s. How far can you get before you're caught and devoured?

2 Funny, It Worked Last Time…

You are in the kitchen below. Half black squares are mirrored surfaces, and raptors may run on clear areas. If a **raptor** sees you in the corner of its eye, it will turn and give chase. Otherwise it will run forward. Raptors are intelligent and do not run into walls. If there is a choice they turn left. There are a number of **bear traps** scattered in the kitchen. If a raptor runs into anything, they're incapacitated and become someone else's problem. Do you survive?



3 What Are The Civilian Applications?

Microraptors are quite like Raptors physically, except that they are smaller and less cantankerous (and are satisfied with Tuna). Being cheap to hire, they can improve a firm's profit margin, and are being rolled out in accountancy firms across the country. It is hoped that no one will notice. Their limiting factor is that they cannot use computers, and instead use the abacus, making use of a wall chart of all possible multiplications of the numbers from 1 to 10 (an old fashioned multiplication table). A Cambridge mathmo visited the office one day, and remarked that the sum of the entries in the table was a perfect square (which was true) and that the sum was 2420. Make a conjecture about the anatomy of Microraptors.

4 The Precise Nature of the Catastrophe

There has been a Zombie outbreak on the Pirate island. Each Zombie infects 1 Pirate per day, and are invulnerable to Pirates. The Pirates have some caged Raptors. Raptors kill Pirates and Zombies at a rate of 1 (Pirate or Zombie)/Raptor/day, and breed at a rate 1 Raptor/Raptor/day. Zombies and Pirates both kill Raptors at a rate of 1 Raptor/(Pirate or Zombie)/day. In line with conventional wisdom, take the continuous limit of Zombies, Pirates and Raptors.

There are P Pirates and $2/9$ P Zombies. The plan, whilst falling outside the normal moral constraints, is to release some of the Raptors, so that the Raptors and Zombies will kill each other before the pirates. How many should be released? It may help to know that according to one congenital optimist, it should be possible to quell the zombie outbreak in $\ln(3)$ days.

5 Another Victim Of The Ambient Morality

You and a law student are in the CMS. In front of you are two choices - one of the rooms contains the answers to Examples Sheet 4 and the other contains your supervisor's pet raptor.

Guarding the way are 3 PhD students. The Statistician always lies, the Pure Mathematician always tells the truth, and the Applied mathmo stabs people who ask tricky questions.

Fortunately, the law student is a potential sacrificial victim. What is the minimum number of questions you need to ask to determine which room contains the solutions, and what is the probability of the lawyer dying under this strategy?

This is the 2011 **Archimedeans Problems Drive**. Some of the problems have tangential relations to mathematics and some were not invented here.

6 Me, I'm Counting

A specific section of the Pirate Island contains 42 cages, each of which can hold precisely one dinosaur. Each dinosaur is either a Raptor, a T-rex or a Dilophosaurus. A particular ordering of dinosaurs is called an arrangement. Determine whether there are more arrangements which contain an even number of Raptors or more arrangements which contain an odd number of Raptors. Note that 0 is an even number.

7 Now We Try It My Way

How much could the Earth's rotation be slowed by the Earth's population attempting to spin on the spot? Make any simplifying assumptions you like. (Note that there is a grey area here, and if the markers disapprove of your assumptions, any excuses and accusations of bias will be given precisely as much credence as in tripos.)

8 Experiencing A Significant Gravitas Shortfall

How likely you are to survive a raptor attack in the following objects:

- CMS Core,
- B pavilion,
- the INU,
- the UL,
- an ACME Klein Bottle?

Explain your answers!

9 Just Read The Instructions

The diagram below shows a series of Raptor pens, which are separated from each other by electric fences.

5	9	4	3	7	2	6
7	3	8		8	2	1
2	9	2	4	7	3	4

Each pen holds a group of raptors; the numbers give the number of raptors in each family. At each stage, the following occurs:

i) The largest group of raptors wakes up.

ii) This group attacks the neighbouring pen which contains the fewest raptors. As they lack tactical grace, one member of the group dies destroying the electric fence.

iii) Killing time: The two groups fight, and an equal number of raptors from each side is destroyed, until there is only one group left.

iv) The surviving group go back to sleep.

This continues until all raptor attacks cease. How many raptors are left alive, and from which pens did they originate?

12 Ultimate Ship The Second

In this question, everyone else is your unwitting accomplice. There are two boxes. Box A might contain 10 points, and box B might contain 5. Alternatively they might contain Raptors, which do not have a point value. Each team is picking a box, and the box that the majority pick will contain a Raptor. The box that a minority pick will contain the relevant number of points. In the event of a tie, both boxes contain points. Choose a box. (Hint: Use Psychology)

10 It's Character Forming

In the interests of promoting science and mathematics to new generations of schoolchildren (and blind to any credibility problem), a new set of standard measurements have been defined: The (male) African Elephant, The Olympic Swimming Pool and the blink of an eye.

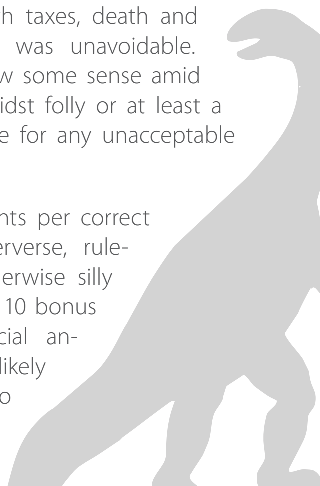
How many kettles would be required to consume one unit of power?

11 Sleeper Service

In the far future, the good ship Arbitrary is carrying a payload of raptors. For reasons of volume, (and to avoid pesky constraints like the speed of light), it is storing them in a cage in hyperspace. Hyperspace has 7 dimensions, and the cage is just large enough for two raptors to be adjacent in each dimension. However, the raptors haven't been happy since the trip started, and so having a pair adjacent would result in a frank exchange of views, some mess, and a tragic loss of cargo. How many raptors can be put in the cage?

Well, you've sat through an hour of lucid nonsense. As with taxes, death and gravity, the end was unavoidable. Hopefully you saw some sense amid madness, wit amidst folly or at least a reasonable excuse for any unacceptable behaviour.

There are 10 points per correct question but perverse, rule-exploiting or otherwise silly answers may get 10 bonus points. The official answers, and unlikely explanations to back them up, can be found on page 92.



Bayesian Bets

Andrew Pollard, Churchill College

The Monty Hall problem is a famous problem in probability: imagine you are on a game show, and you need to choose one of three doors. Behind one of the doors is the star prize, a car, and behind each of the other two is a relatively disappointing goat. Once you have chosen a door, the game show host then opens one of the other doors to reveal a goat. You then have the opportunity to change your mind and choose the other remaining door if you wish. Should you do so?

The Bayesian Approach

Most people think that it doesn't matter, and some people think that you should switch (for example, almost everyone reading this article). Anyone who has studied some probability will tell you that the probability of winning the car if you switch is 2/3, but everyone else will insist that it's 50:50. When faced with the Monty Hall problem, most people think of the position of the car as random, which suggests that most people are intuitively Bayesian. Let's have a look at what this means.

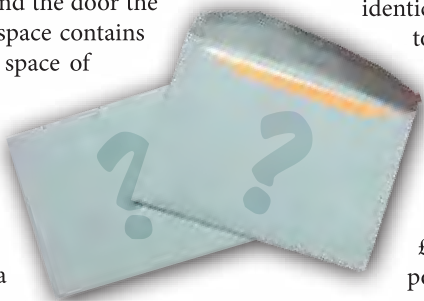
Two of the main approaches to problems such as the Monty Hall problem are the *frequentist approach* and the *Bayesian approach*. In the frequentist approach, the position of the car is regarded as an *unknown*, but non-random, parameter; in this setting it is often easier to apply basic probability theory. In the Bayesian approach, the position of the car is regarded as a *random* parameter, Θ, to which is assigned a specified *prior distribution*, which is subjective; it is a (hopefully educated) guess at how Θ behaves. When data is observed, one updates the prior distribution to take the data into account, forming the *posterior distribution* for Θ, from which we can make inference about Θ. A sensible prior distribution for the position of the car would be the uniform distribution, and this would lead to a posterior agreeing with the frequentist result.

Bayesian methods are popular because of their ability to incorporate new information and update statistical models.

Decision-making

If we wish to make decisions using a Bayesian approach, we must also specify a *loss function*. If a is some act, the loss $L(\theta, a)$ measures the loss from taking the act a when our parameter Θ takes the value θ. For example, we might use *zero-one loss* in the situation where some acts will result in success and the others in failure. In that case we assign 0 to an act resulting in success, and 1 to an act resulting in failure. For more complicated (often continuous) situations, it may be appropriate to use a more complicated loss function.

For the Monty Hall problem, it would seem sensible to use 0–1 loss, unless you would like a goat. So we have a uniform prior distribution, and we have the 0–1 loss function. To make decisions, we have to look at the *risk function*. Let $\mathbb{X}$ be the sample space, i.e. the set which our data takes values in (in the Monty Hall case, our data are the door we have chosen and the door the host shows us, so the sample space contains pairs of doors). Let $\mathbb{A}$ be the space of all possible acts, and define a (non-randomised) *decision rule* as a function $d : \mathbb{X} \to \mathbb{A}$. Let $\mathbb{D}^0$ be the set of all non-randomised decision rules. Then we can define a general (randomised) decision rule as a random variable D taking values in $\mathbb{D}^0$ with some known distribution Δ. We can also define the *risk function* of a decision rule with $d \sim \Delta$ as



$$R(\theta, d) = \mathbb{E}_{d\sim\Delta}\Big[\mathbb{E}_\theta\big\{L\big(\theta, d(X)\big)\big\}\Big].$$

The Monty Hall problem is a case where the set $\mathbb{D}$ of all randomised decision rules is easily identifiable:

$$\mathbb{D} = \big\{d = \{\text{stick with probability } p, \\ \text{switch with probability } 1-p\} : p \in [0,1]\big\}.$$

Thus we can calculate the risk function of an arbitrary decision rule with respect to 0-1 loss:

$$R(\theta, d) = p\,\mathbb{E}_\theta\big\{L(\theta, \text{stick})\big\} \\ + (1-p)\,\mathbb{E}_\theta\big\{L(\theta, \text{switch})\big\}.$$

Now

$$\mathbb{E}_\theta\big\{L(\theta, \text{stick})\big\} = \mathbb{P}_\theta(\text{lose after stick}) = 2/3,$$

and

$$\mathbb{E}_\theta\big\{L(\theta, \text{switch})\big\} = \mathbb{P}_\theta(\text{lose after switch}) = 1/3,$$

so we deduce $R(\theta, d) = (1+p)/3$. Note that the risk is independent of θ, so is uniformly minimised when $p = 0$, i.e. the "always switch" rule. A decision rule δ is called *Bayes* with respect to a given distribution Π for Θ if it minimises the expected risk. This is written as

$$\delta = \arg\min_{d\in\mathbb{D}} \mathbb{E}_{\Theta\sim\Pi}\big\{R(\theta, d)\big\}.$$

Since, in the Monty Hall case, the risk function is independent of θ, the switching rule is Bayes with respect to *any* distribution for Θ.

By the way, Monty Hall presented a game show called *"Let's Make a Deal"*, but this game was never actually on it!

The Two Envelopes Problem

Now that we are acquainted with the frequentist and Bayesian analyses, let's consider another game show. You are shown two envelopes, identical in appearance, and you are told that each contains money: one of the envelopes has twice the amount of money as the other, but you don't know which. You pick one at random, and you open it to find that it contains, say, £20,000. You then have the opportunity to switch envelopes if you wish.

From a frequentist perspective, suppose the smaller amount of money is some fixed (but unknown), non-random parameter m. Then it is your guess that provides the randomness, with a 50:50 chance of picking the envelope with m pounds in. In this case the fact that your envelope contains £20,000 tells you nothing; it is still a 50:50 chance that you would do better by switching.

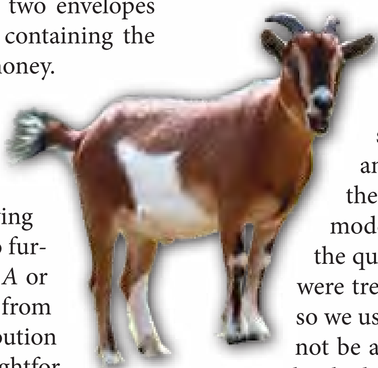
As an interesting exercise, suppose for a moment that you do not open the envelope. Let's call the unknown amount in your envelope M. Then reason thus: the expected amount in the other envelope is $\frac{1}{2} \times \frac{1}{2}M + \frac{1}{2} \times 2M = \frac{5}{4}M$, so this suggests you should switch. But then we can repeat the argument again, calling the amount in the other envelope N, and so we should switch again! We thus have a strategy where we never settle on an envelope. What went wrong? We went from treating the amount in our envelope as non-random and the amount in the other envelope as random

Get as close to a target number using six integers and arithmetic operation in the British game show *Countdown*.

▼ You can win a car or a goat.
Armin Kübelbeck

to the other way round – it's important to be consistent!

Let us try the Bayesian approach, constructing the problem more carefully. Call the two envelopes A and B. Let Θ be the envelope containing the (unknown) smaller amount of money. Give Θ a uniform prior distribution on $\{A,B\}$. Now let X be your chosen envelope, also with a uniform distribution on $\{A,B\}$ (independently of Θ). In observing that X contains £20,000, we are no further to determining whether $X = A$ or B. Thus we infer nothing about Θ from this observation, and our distribution for Θ remains the same. It is straightforward to show that under this joint distribution for X and Θ, $\mathbb{P}(\Theta = X) = 1/2$. So far this is looking similar to the frequentist analysis.



Money is better than goats

It's when we introduce a loss function that things start to become more variable. The value of different amounts of money is different for each person. Whichever way you look at it, £20,000 is pretty good – better than a goat at any rate, so we shouldn't treat it the same! Let's assign a loss of ℓ_1 for getting £10,000 and h_1 for getting £20,000 *when the other envelope contained £10,000* – notice that we might not wish to set $h_1 = 0$ if we want to represent how much better the higher amount is than the lower amount; the ratio between h_1 and ℓ_1 could model just how much we care about the quantities. With cars and goats we were treating the car as infinitely better, so we used zero-one loss, but that might not be appropriate here. Similarly let ℓ_2 be the loss for getting £20,000 when the other envelope contained £40,000, and h_2 the loss for getting £40,000.

The decision space $\mathbb{D}$ for this problem is the same as for the Monty Hall problem. Consider the rule

$$d = \{\text{stick with probability } p,\ \text{switch with probability } 1-p\}.$$

Then the risk function is

Can Mathematics help succeed in the famous game show *Who want's to be a Millionaire?*
Neon Circus Ltd

$$R(\theta,d) = p\,\mathbb{E}_\theta\{L(\theta,\text{stick})\} + (1-p)\,\mathbb{E}_\theta\{L(\theta,\text{switch})\}.$$

Now

$$\mathbb{E}_\theta\{L(\theta,\text{stick})\} = \mathbb{E}_\theta\{L(\theta,\text{stick}) \mid X=\theta\}\,\mathbb{P}_\theta(X=\theta) + \mathbb{E}_\theta\{L(\theta,\text{stick}) \mid X\neq\theta\}\,\mathbb{P}_\theta(X\neq\theta) = \tfrac{1}{2}(\ell_2+h_1).$$

Similarly

$$\mathbb{E}_\theta\{L(\theta,\text{switch})\} = \tfrac{1}{2}(\ell_1+h_2),$$

so the risk is

$$R(\theta,d) = \tfrac{1}{2}\big(p(\ell_2+h_1)+(1-p)(\ell_1+h_2)\big).$$

Thus we see that finding a good rule in this case depends heavily on the subjective values ℓ_i, h_i. Notice that the rule $p=0$, i.e. always switching, is Bayes iff $\ell_2-\ell_1+h_1-h_2\geq 0$, and uniquely Bayes if the inequality is strict, whereas the rule $p=1$, never switching, is Bayes and uniquely Bayes for the corresponding reversed inequalities. For example if we choose $h_1=h_2=0$, $\ell_1=10\,000$ and $\ell_2=20\,000$, then $\ell_2-\ell_1+h_1-h_2=10\,000>0$, so switching is the unique Bayes rule. If however we take the loss from getting m pounds to be $10\,000/m$ we get $\ell_2-\ell_1+h_1-h_2=-1/4<0$, so sticking is the unique Bayes rule.

Is that your final answer?

Try analysing the games *Who Wants to Be a Millionaire?* and *Deal Or No Deal* using frequentist and Bayesian approaches – taking into account the value of each question or box might make things interesting. After using a 50:50 in *Who Wants to Be a Millionaire?* (the computer removes two of the three wrong answers at random), it is *not* advantageous (under frequentist analysis) to switch guesses. Can you see why this is different to the Monty Hall problem? Is there a Bayes rule for every situation in *Deal or No Deal*? Is that your final answer?

References, Further Reading

1. A. P. Dawid, *Principles of Statistics Notes*, Cambridge Mathematical Tripos Part II
2. R. Eastaway, J. Wyndham, *How Long is a Piece of String?*, Chapter 5: Should I Phone a Friend?

Fractal Finance

Aidan Chan, Peterhouse College

How useful is financial mathematics in the real world? It depends on the underlying assumptions, the data input and the judgement of the user (and possibly more). Nonetheless, there were a lot of recriminations and accusations that mathematical finance had led the world into the financial crisis of 2007 – 2010. Of course, there were more perpetrators than the religious orders of high finance, namely the US government which encouraged people to own homes they couldn't afford, bad incentives, and cheap capital from China.

In the last 30 years, finance and economics have tried to imitate physics, placing an emphasis on the use of mathematics and theorem-proof structures, which requires assumptions. This is a tricky business: in finance, we are in a largely modelling human behaviour. The laws of physics are accurate to ten decimal places (at least classical physics); we can predict physical behaviour reliably. Not so in finance. People are irrational, being influenced by events, their own feelings (swinging between greed and fear rather abruptly), and their expectations of other people's feelings. Combined with the complex physical world of weather, epidemics, crops, ores, and factories, modelling becomes considerably harder. The economy is a *complex adaptive system*. Empirically, the financial markets have complex, not binary payoffs, and the underlying probability distributions are fat-tailed. This is called Quadrant IV by Nassim Nicholas Taleb, and this is where conventional statistics fails us.

> *"…what's brought the global banking system to its knees isn't simply greed or wickedness, but – and this is far more frightening – intellectual hubris."*
>
> **John Gray**

Underlying Probability Distribution	Payoff	
	Simple (Binary)	Complex
Mild	I (safe)	II (Safe)
Wild	III (safe)	IV (dangerous)

In economic life, which sits in Quadrant IV, empirical evidence shows that a laundry list of assumptions used in mathematical models are wrong: markets are not continuous, volatility is not constant, previously uncorrelated markets can start to move together (correlation risk).

The world is Mandelbrotian

The father of fractal geometry, Benoit Mandelbrot, had a predilection for the markets. He distinguished two types of randomness: mild (Gaussian) and wild (power laws). According to the former, market price changes can be picked at random from a pile consisting of sand grains of different sizes. In the latter, price changes are picked from a pile, containing dust, sand, stones, rocks, and boulders. Power-law distributions have higher moments that are unstable or changing over time, and for these distributions, the central limit theorem fails. Wild randomness can have one event dominating the rest; mild randomness typically doesn't.

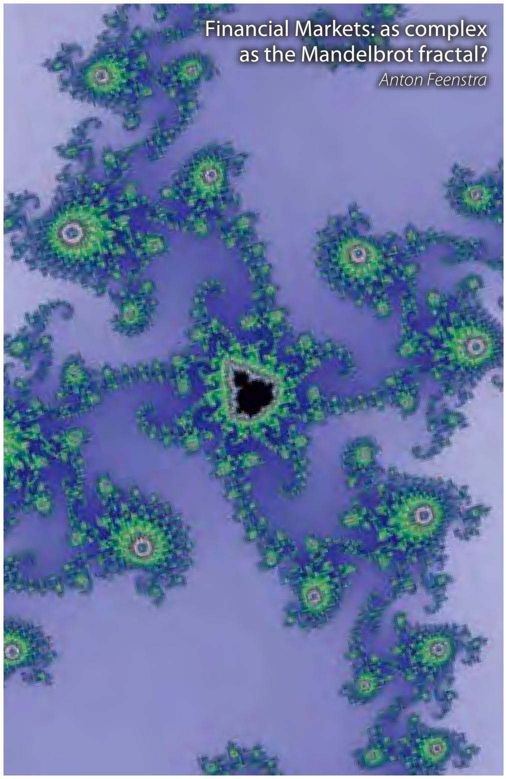
Sum of the first three squares, i.e. a *square- pyramidal number*. No, we didn't make up this name!

The frequency of an event follows a power law when it varies as a power of some attribute of that event. Since power laws exhibit scale invariance, an easy way to understand power-law or scaling systems is using conditional probability. For example, if the probability of a loss of 1000 given a loss of 100 are the same as that of a loss of 10 000 given 1000, that's a power law. The most famous is Pareto's power law. He found that roughly, 20% of Italians owned 80% of all the land in his time. Within that 20%, again 20% of the 20% own 80% of the 80% of the land, i.e. 4% own 64% of the land.

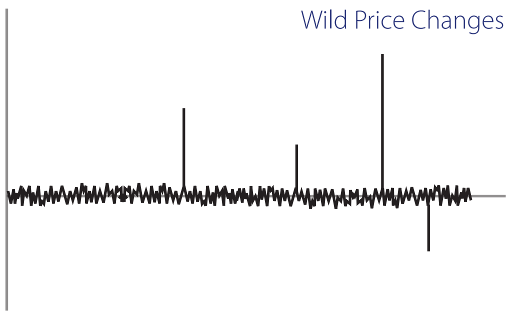
Most classical applications of statistics are based on the key assumption that the data distribution is Gaussian, or some other known form. Classical statistics work well and allow you to draw precise conclusions if you're correct in your assumption of the data distribution. However, if your distribution assumptions are even a little bit off, the error is enough to derail the delicate statistical estimators. Mandelbrot's conjecture: price change distributions have infinite variance; sample variance (the implied variability in prices) simply increases as more data is added. If this were true, most standard statistical techniques would be invalid for price data applications. Unfortunately, there are statistical problems in determining if the variance of price change is infinite. Gathering enough data to "assure" that price change variance is infinite might take a century. But, if market prices have infinite variance, any classically derived estimate of risk will be significantly understated.

Tomorrow doesn't look like yesterday

Recall Bertrand Russell's turkey (or induction) problem. In markets: *"this event has never happened in my market"*. Suppose one were living in the era of September 1987, just before the financial crisis in October that year. The worst change in markets on any given day, based on historical data before 1987, was –10%. So models were mostly calibrated to simulate what happened to the portfolio, based on the worst case scenario of a –10% change on a given day. And sure enough, there was a –23% move in the markets in one day in October 1987. Models may be recalibrated such that the worst-case scenario is –23% in a day, but the next time it may be a –50% move. Data from the past may not be relevant to today, again because of complexification of the economy. They are still looking through the rearview mirror and they have by definition very few data points in that



Financial Markets: as complex as the Mandelbrot fractal?
Anton Feenstra



region. In fact, for some markets, one observation will account for 80%, even 90%, of the "weight" in the estimation of the magnitude of the tail risk.

Complexification of the economy means that rare events are increasingly unpredictable, with consequences that are increasingly dire. If someone says that this extreme event, according to their mathematical model, occurs no more than once in 10 000 years, and if they have not lived 10 000 years, it is fair to assume they are not drawing their conclusion from their own empirical experience but from some theoretical model that produces the risk of rare events. More often than not, the model is wrong about rare events.

Consider also the self-reference problem, pervasive in financial mathematics: when do we have enough historical data to make an inference about the probability distribution? If the distribution is Gaussian, then we are able to say we have enough data – the normal distribution itself tells us how much data we need. But if the distribution is not from a well-behaved family, we may not have enough data. But how do we know which distribution we have? From the data itself! If one needs data to obtain a probability distribution to approximate knowledge about the future behaviour of the distribution from its past results, and if, at the same time, one needs a probability distribution to determine data sufficiency and whether or not it is predictive outside its sample, we have a severe self-reference problem and have no idea what weight to put on additional data points.

How much can you lose?

The most popular model being used by financial firms is Value-at-Risk (VaR). Normal, parametric VaR modelling is based on assumptions. Three false ones: stationary (constant-shaped probability distribution over time), Brownian motion/random walk (tomorrow's outcome is independent of today's outcome) and normally-distributed price changes. VaR works as such: a financial firm decides how "safe" it wants to be. Say, it sets a 99% confidence level. Its investments are ostensibly structured so that there is only a 1% chance of breaking through the danger point. The model also inputs an array of variables, including diversification, leverage and volatility, to calculate the market risk. With a few more strokes, a risk manager can get an answer, e.g. that his firm's portfolio has a 1% chance of losing more than £50 million (or say, 20%) this week.

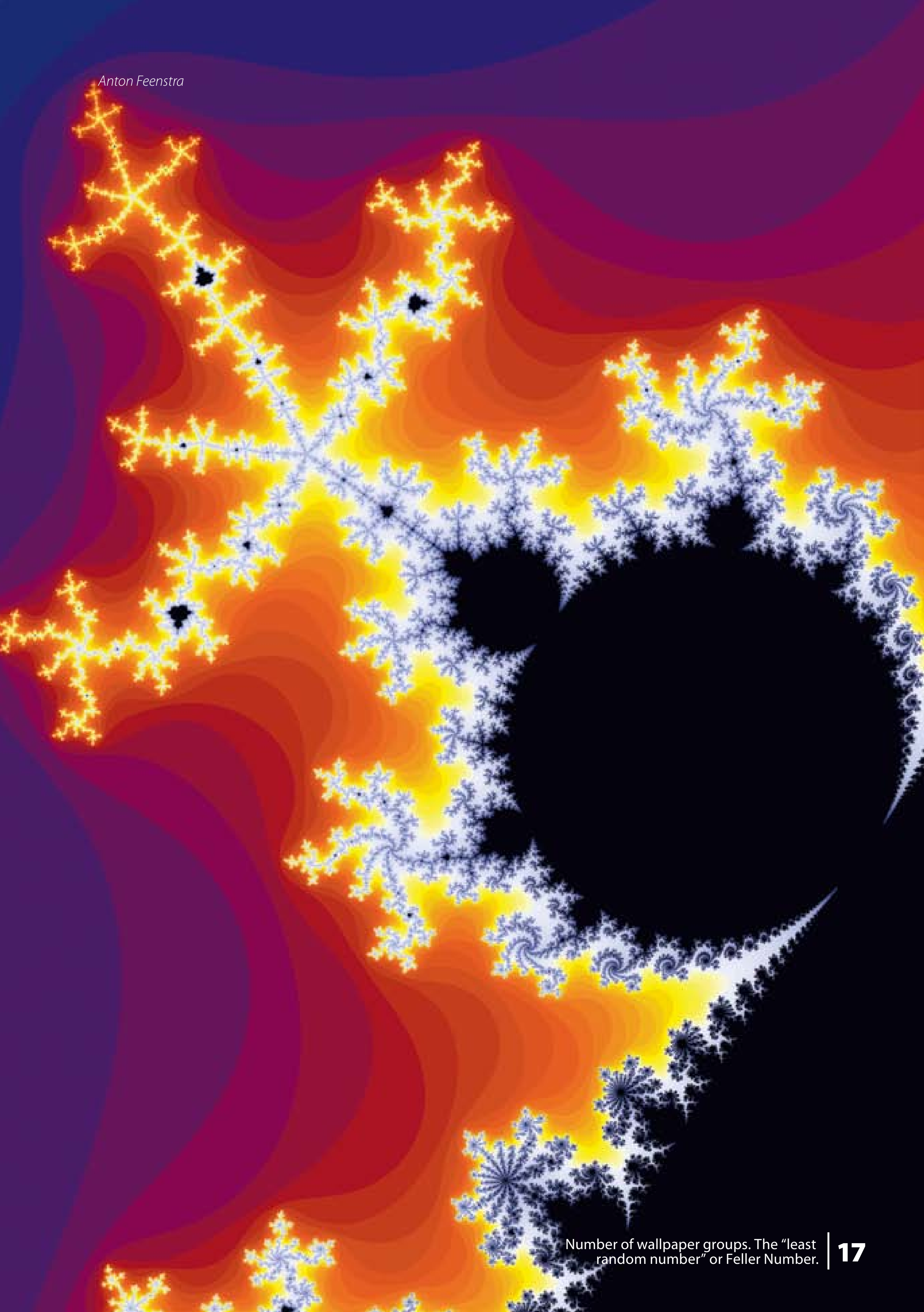
The weakness of VaR seems to be that it measures *boundary risk* instead of *expected value*. It doesn't really answer the question "how much can I lose?". If one were at a casino, and was offered an exotic game with no entrance fee, where 99% of the time one wins £10, and 1% of the time one loses £1,000,000, it would be wise not to play, for the game has a negative expected value (here assuming one is not playing a game like the *St. Petersburg Paradox*).

One fixture to the VaR model is Extreme Value Theory, which is gaining popularity. It assumes price changes scale, and that there are "fat tails". However, it does not fix the problem of long-term dependence, which is the scenario where bad news is followed by more bad news; a day of down in the markets is more likely to be followed by another day of down. A bank may survive one *Black Swan* event, but not 2 or 3 in succession (e.g. earthquake in Christchurch followed by Fukushima, if the bank is invested in securities exposed to both events).

Options

Options are the right, but not the obligation to buy or sell a certain asset at a certain price at a certain time. The Black-Scholes options pricing formula identified key variables that affect what an option is worth: where the price of the asset currently is, compared to the strike price (where the price has to be for the bet to show a profit), how volatile the underlying asset tends to be, how much time before expiration, and prevailing returns on risk-free investments.

Such a pricing formula had in fact been used by traders in Chicago years before and was based on the principles developed by mathematician and gambler Ed Thorp. Options had been actively trading at least in 1600 as described by De La Vega (1688), implying a heuristic method to price them and deal with their exposure. What Merton and Scholes did was to make it aligned with financial economic theory, by deriving it assuming "dynamic hedging", a method of continuous adjustment of portfolios by buying and selling securities in response to price variations. Dynamic hedging assumes no jumps; it fails miserably in markets and did so catastrophically in



Anton Feenstra

Number of wallpaper groups. The "least random number" or Feller Number.

1987. Black-Scholes should be used merely as a pricing guide for options, since we know volatility is not constant and markets jump. Funds that employ high leverage to bet on arbitraging options' market price and Black-Scholes price is liable to blow up, Long Term Capital Management-style, when markets suddenly turn rough.

Escaping Quadrant IV

Other modelling methods relying on variance as a measure of dispersion, Gaussian Copulas, and the ARCH family of models are incapable of prediction where fat-tailed distributions are concerned. Part of the problem is that these methods miscalculate higher statistical moments (for non-linear magnitudes such as highly-leveraged reinsurance, higher moments are important), and thus lead to catastrophic estimation errors. No method will work for more than a short time horizon, just as no weather forecast works well over a period of 2 weeks.

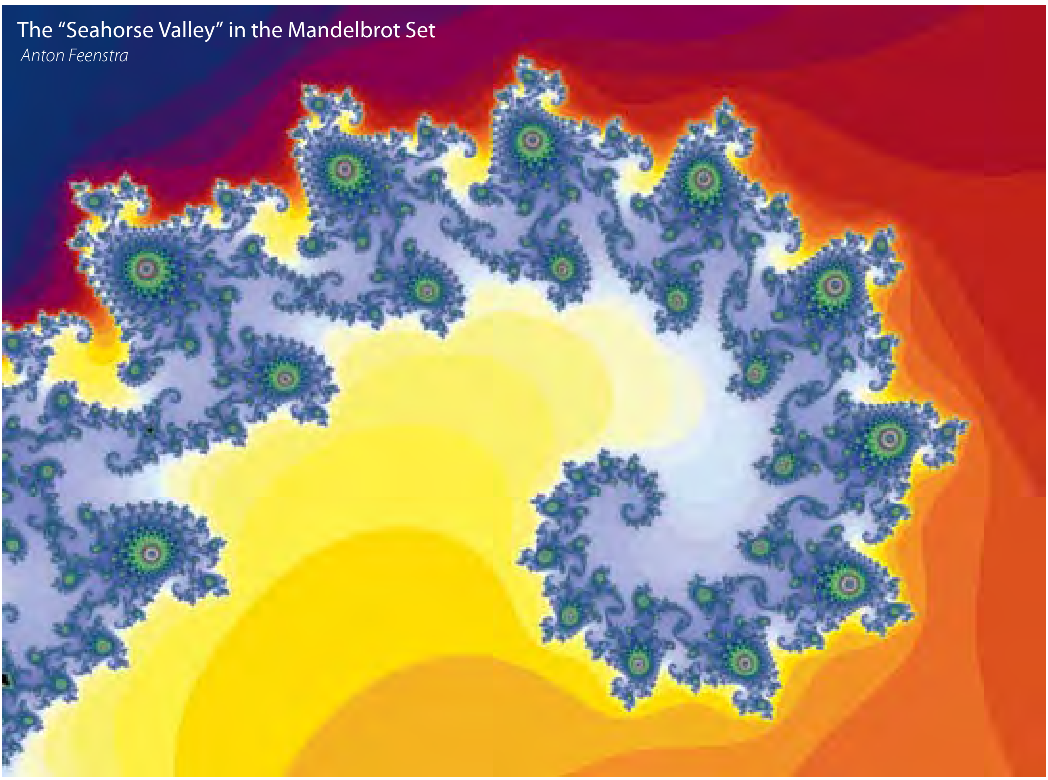
To make our forecasts more reliable, we should escape Quadrant IV. We can attempt this through payoff truncation (for an insurer: reinsurance and payoff maximums; for an options trader: don't sell naked options), ostensibly swapping complex payoffs to simple payoffs (reducing leverage); in reality we are still in Quadrant IV. Counterparty risk is still in play.

Perhaps it's better is to try not to predict; Mandelbrotian fractal models based on power laws can help us understand better the behaviour of *Black Swans*, although they do not help to forecast.

The World is Complex

Forecasting is becoming less and less reliable because our world is becoming more interdependent, and complexification has only increased since 1995 because of the internet. You do not have the independence assumption used in many models before the crisis.

We are safer using much larger data samples over much longer time periods to form our judgements, while actively searching for counterexamples to our initial results. Over-reliance on modelling is a severe limitation. One must combine good judgement and experience together with the use of



The "Seahorse Valley" in the Mandelbrot Set
Anton Feenstra

Only non-zero number that is twice the sum of its digits. Age of majority in most countries.

models such as VaR. Without models and mathematics, it would be hard think about finance and economics, but it's important to fit our models to the world, instead of expecting the world to obey our models. A tailor makes a suit for a client by measuring him and cutting the cloth to fit, not by performing surgery on him. The use of any financial model should entail us questioning where it can go wrong, and how practical it is despite its assumptions. We need robust statistical estimators, estimators which are not perturbed much by mistaken assumptions about the nature of the distribution.

Finally, we should heed the modeller's Hippocratic Oath by Emanuel Derman and Paul Wilmott:

- I will remember that I didn't make the world, and it doesn't satisfy my equations.
- Though I will use models boldly to estimate value, I will not be overly impressed by mathematics.
- I will never sacrifice reality for elegance without explaining why I have done so.
- Nor will I give the people who use my model false comfort about its accuracy. Instead, I will make explicit its assumptions and oversights.
- I understand that my work may have enormous effects on society and the economy, many of them beyond my comprehension.

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These five interlocking tetrahedra are made from 30 individual pieces, using no scissors or glue.
Philipp Legner

Number of rooted trees with 6 vertices. Number of faces of an icosahedron and vertices of a dodecahedron.

Mathematical Origami

Philipp Legner, St John's College

Origami shapes and paper models of Archimedean solids are not only nice to look at, they give rise to an interesting area of mathematics. Similarly to the idea of *constructing* polygons using nothing but a straight edge and a compass, you can think about which shapes and solids you can *fold* using a sheet of paper and no other tools. The results are surprisingly different from ruler and compass geometry!

Even more beautiful objects can be created if you *are* allowed to use scissors and glue. I have included photos and folding patterns of braided platonic solids, knotted pentagons and interlocking polyhedra.

The Axioms of Origami

In 1992, the Italian-Japanese mathematician Humiaki Huzita published a list of all possible operations that are possible when folding paper.

O1 We can fold a line connecting any two points P and Q.

O2 We can fold any two points onto each other.

O3 We can fold any two lines onto each other.

O4 Given a point P and a line L, we can make a fold perpendicular to L passing through P.

O5 Given two points P and Q and a line L, we can make a fold that passes through P and places Q onto L.

O6 Given two points P and Q and two lines K and L, we can make a fold that places P onto line K and places Q onto line L.

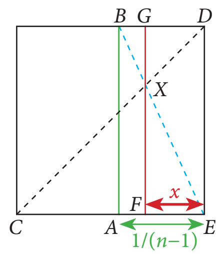
A seventh one was discovered by Koshiro Hatori:

O7 Given a point P and two lines K and L, we can fold a line perpendicular to K placing P onto L.

This set of axioms is much more powerful than the one corresponding to straight edge and compass: connecting any two points with a straight line and drawing a circle of radius r around any point. There are many interesting consequences: you can trisect angles, double cubes and even construct regular heptagons and 19-gons.

Even more surprisingly, we can use Origami to fold *any* rational number. Consider a square piece of paper of side length 1 and suppose, for induction, that we can fold one side into $(n-1)$th, as shown. Then we can also fold it into nth the by

- folding the along **CD**;
- folding the line **EB**;
- folding the line **FG** through X, perpendicular to the edge of the paper.



Now observe that

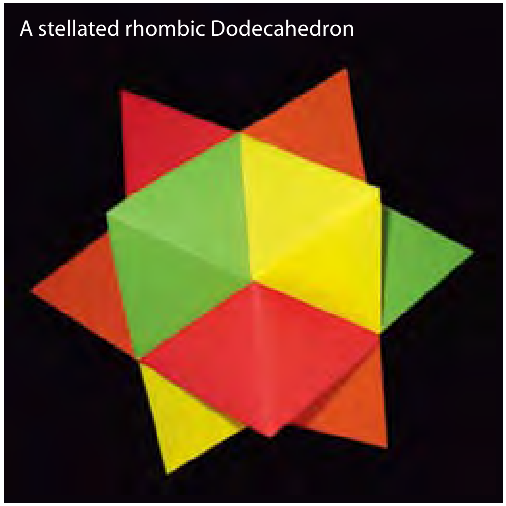
$$\frac{\frac{1}{n-1}}{x} = \frac{|AE|}{|FE|} = \frac{|AB|}{|FX|} = \frac{1}{1-x}$$

$$1 - x = x\,(n-1)$$

$$x = 1/n,$$

so we have divided the side of the square into nth. We can easily divide the side of the square into halves. Thus, by induction, we can use origami to fold any ratio, as required.

One can make a perfect regular pentagon simply by knotting a strip of paper.
All by Philipp Legner



A stellated rhombic Dodecahedron

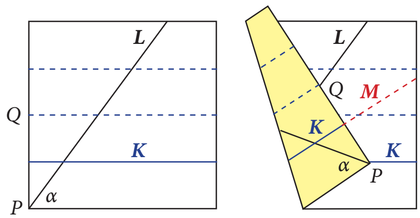
Cunning Constructions

The proofs of the following constructions are left to the reader. They are based on simple geometric relations and can also be found in [3].

Trisecting the Angel

We start with a square piece of paper and fold a crease L, as shown below, to create any angle α at P. To trisect α, we have to

- fold the paper into **quarters** from top to bottom and define K and Q as shown;
- simultaneously fold P onto K and Q onto L using axiom 6, and don't reopen;
- extend K by a new crease M.

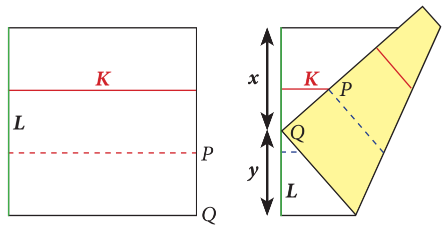


If we now open the paper and extend M to its full length, it will divide α in the ration 1:2. Halving the larger part of the angle then splits α into three equal parts.

Doubling the Cube

Even the ancient Greeks knew that it is impossible to double a cube, i.e. construct $\sqrt[3]{2}$ using nothing but ruler and compass. It was rather discouraging when the oracle in Delphi prophesied that a plague could be defeated by doubling the size of the altar to Apollo – if only they had known ancient Japanese Origami artists…

Again let us start with a square sheet of paper. We first fold the paper into thirds (since we can fold any ratios), and define K, L, P and Q as shown below. We now fold P onto K and Q onto L using axiom 6, and the ratio of the lengths x and y in the diagram is precisely $\sqrt[3]{2}$.



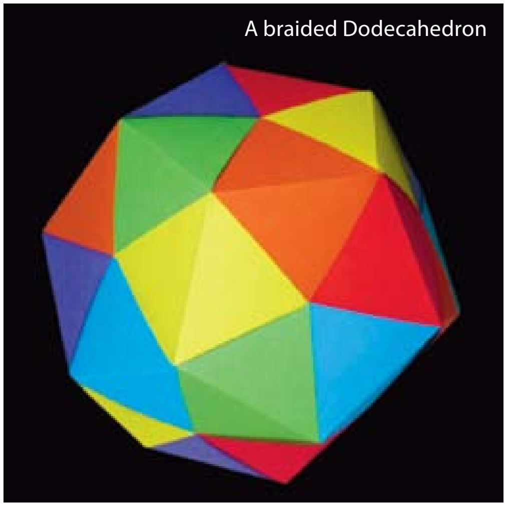
Incidentally, the third "classical" problem that is impossible with straight edge and compass, squaring the circle, is impossible even using Origami, since it involved constructing the transcendental ratio $\sqrt{\pi}$.

Solving Cubic Equations

It is known that quadratic equations can be solved with straight edge and compass. With Origami, we can also solve *cubic* equations.

Suppose have an equation $x^3 + ax^2 + bx + c = 0$. Let $P = (a,1)$ and $Q = (c,b)$ in a coordinate system. Furthermore, let K be the line $y = -1$ and L be the line $x = -c$ as shown below. Using axiom 6, we can simultaneously place P onto K (at P') and Q onto L (at Q') to create a new line M. Suppose that M has equation $y = \alpha x + \beta$, for some $\alpha, \beta \neq 0$.

An Origami Icosahedron

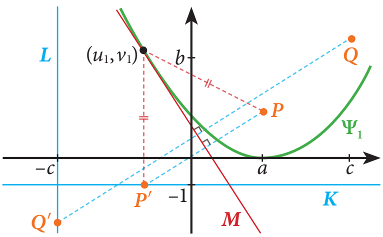


A braided Dodecahedron

Let Ψ_1 be the parabola $4y = (x - a)^2$ with focus P and directrix K. Then M is a tangent to Ψ_1 at a point (u_1, v_1) – this is illustrated by the dotted red lines in the diagram below. Differentiating gives $\alpha = \frac{1}{2}(u_1 - a)$ and we can deduce $\beta = -\alpha^2 - a\alpha$.

Let Ψ_2 be the parabola $4cx = (y - b)^2$ with focus Q and directrix L. Again M is a tangent to Ψ_2 at a point (u_2, v_2) and we can find $\beta = b + c/\alpha$.

Setting these two results equal shows that $\alpha = \frac{1}{2}(u_1 - a)$ satisfies $x^3 + ax^2 + bx + c = 0$, i.e. is the solution we are looking for.



Doubling the cube is equivalent to solving the cubic $x^3 - 2 = 0$, while trisecting the angle is equivalent to solving $x^3 + 3tx^2 - 3x - t = 0$ with $t = 1/\tan\theta$ and $x = \tan(\theta/3 - \pi/2)$.

We can define the set $\mathcal{O}$ of *Origami Numbers*, numbers that can be constructed using origami. It *includes* the corresponding set for straight edge and compass constructions, and is *the same* as for constructions using a market rule and compass.

We can also construct many regular polygons using Origami: precisely those with $2^a 3^b \rho$ sides, where ρ is a product of distinct Pierpont primes, that is, primes of the form $2^u 3^v + 1$.

The Art of Folding Paper

Finally let us look at some of the artistic aspects of paper folding. The word *Origami* (折り紙) originates from the Japanese *oru* (to fold) and *kami* (paper). Japanese monks were among the first to turn the amusement into a sophisticated art. On the next page you can see some great examples.

Especially useful for creating mathematical solids if *modular origami*: you fold many individual pieces (such as faces) separately and then stack them together.

An ingenious method for creating Platonic solids is "braiding" particularly shaped strips of paper. When using differently coloured paper, this produces some of the most beautiful and decorative objects. The patterns can be found on the following page. Start with the tetrahedron before attempting the larger solids – you may need lots of paper clips, or another pair of hands!

References, Further Reading

Robert Lang's website (see below) is a great place to start both for building origami and reading about the mathematical background.

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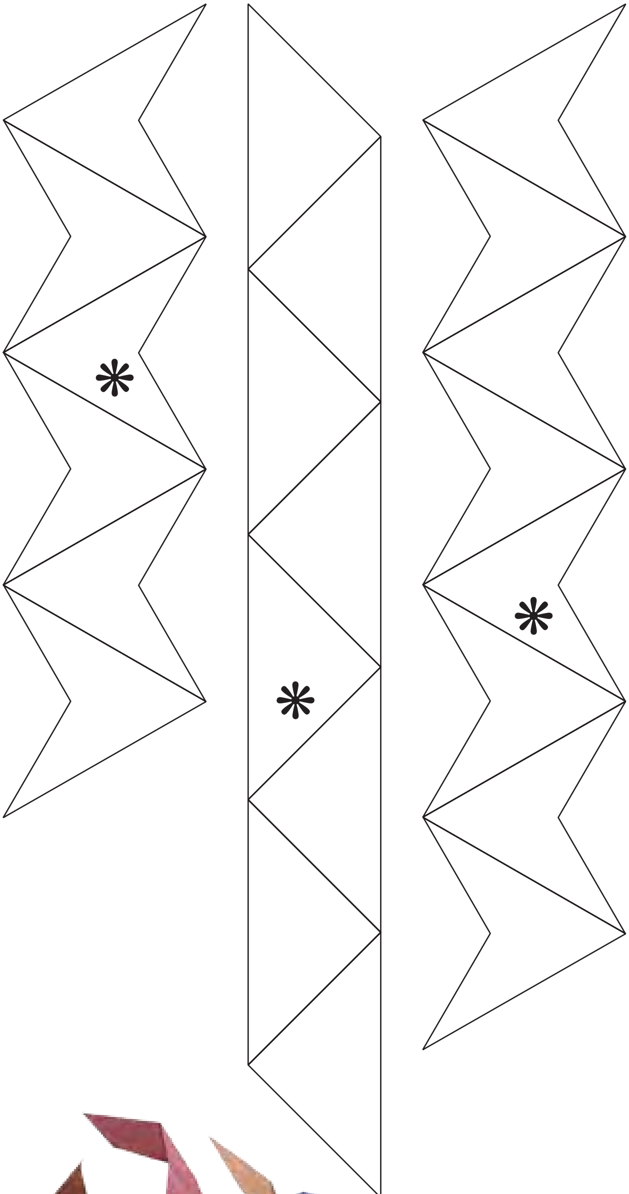
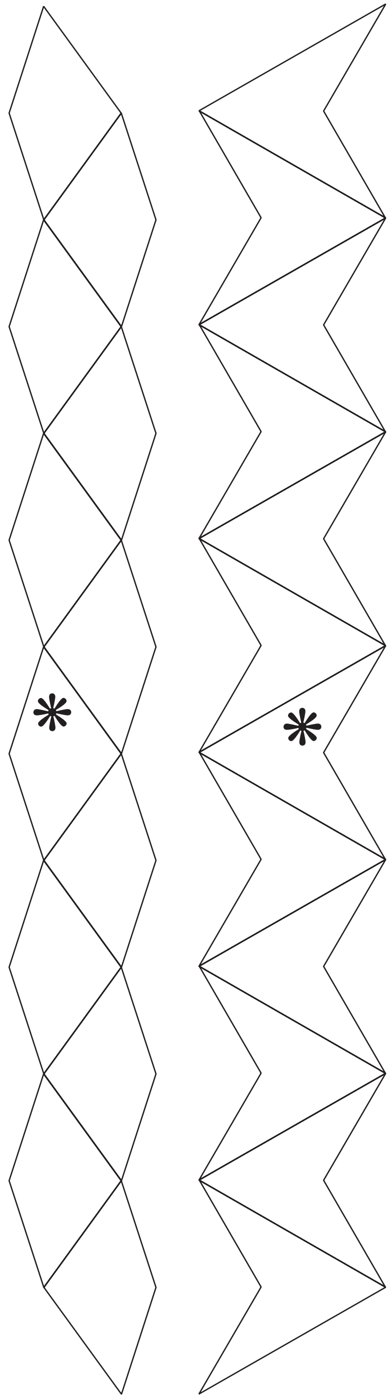
Upupa Epops, Fugu and Crayfish
Sipho Mabona

For best effect, enlarge this page to A3 and copy it on heavy, coloured paper. Start by cutting out the required number of strips for each solid and carefully creasing all lines in the same direction. The stars should be visible on the inside of the bottom face; only for the Dodecahedron and Icosahedron some strips are added later. It will be helpful to use paperclips to hold the finished faces in place.

Tetrahedron	**Cube**	**Octahedron**	**Dodecahedron**	**Icosahedron**
3 needed	4 needed	4 needed	6 needed	6 needed





Quantum Entanglement

Elton Yechao Zhu, Queens' College

In quantum mechanics, a particle does not have a definite state (e.g. position, momentum etc.), but rather can exhibit a few states simultaneously, each with certain probability. The spin of an electron (loosely speaking, the direction of its angular momentum) has two states upon standard basis measurement, up (denoted by $|0\rangle$) and down (denoted by $|1\rangle$). Hence, the quantum state of an electron is a superposition of these two states. $|\phi\rangle = a\,|0\rangle + b\,|1\rangle$, a and b being complex numbers. $|a|^2$ is the probability of getting $|0\rangle$ when a measurement is made and $|b|^2$ is the corresponding one for $|1\rangle$. Similarly, a combined state of two electrons could be $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|1\rangle|0\rangle$, $|1\rangle|1\rangle$ upon standard basis measurement. Any linear combination of these four states with the appropriate coefficient (i.e. the sum of probabilities is 1) is a feasible quantum state.

"I think I can safely say that nobody understands quantum mechanics."

Richard Feynman

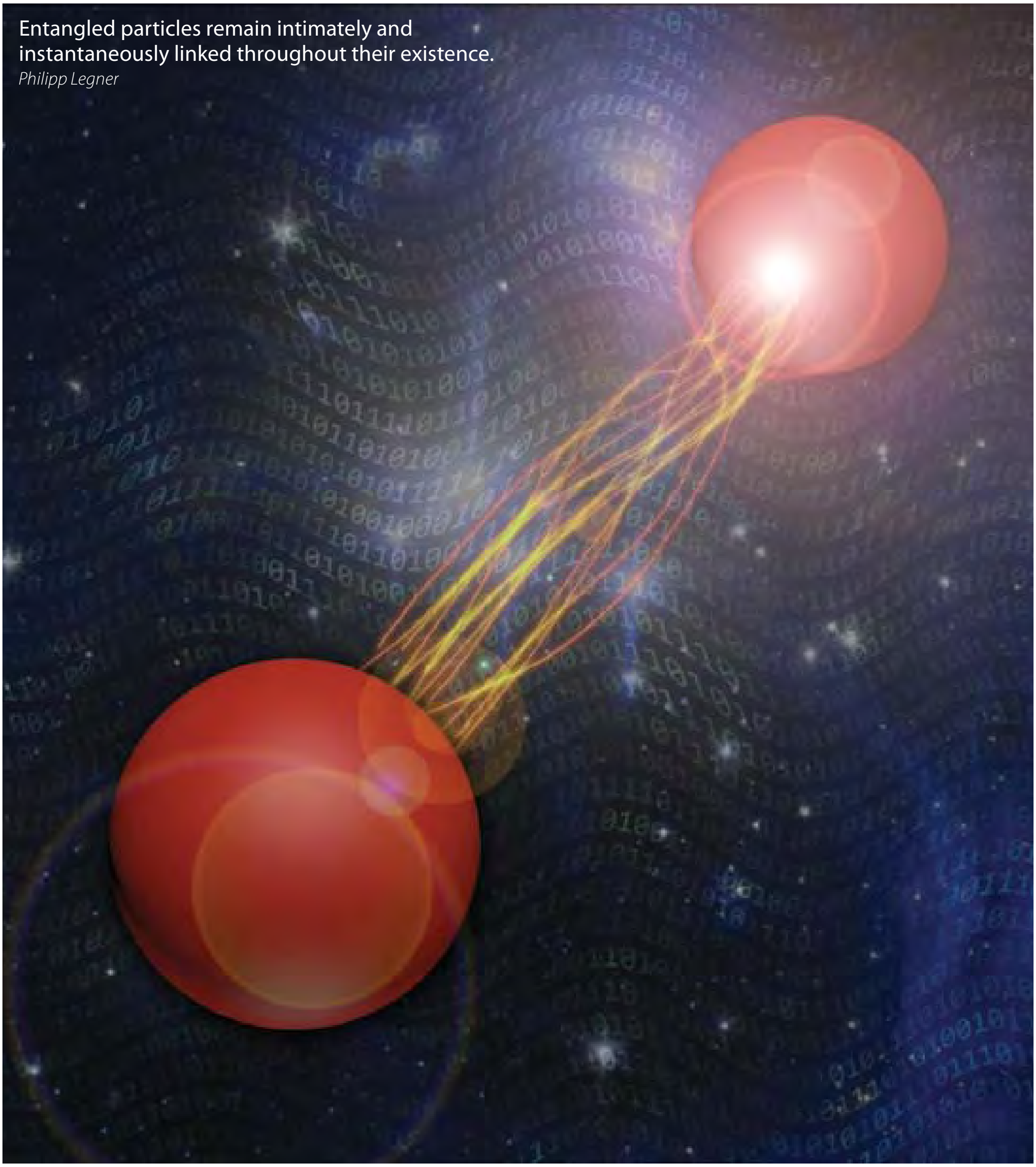
Entangled Quantum States

An entangled state is a quantum state which cannot be written as a product of individual states. For example, $\frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right) \neq |\phi_A\rangle|\phi_B\rangle$ for any possible ϕ_A and ϕ_B (If you don't believe it, please give it a try). Suppose Alice has electron A and Bob has electron B. A and B are entangled with the above state. Alice and Bob are physically separated but can choose to communicate with each other (e.g. a telephone line). If Alice measures electron A and Bob measures electron B instantaneously afterwards, then the state of electron B will always be the same as that of electron A, since their combined state could only be $|0\rangle_A\,|0\rangle_B$ or $|1\rangle_A\,|1\rangle_B$. This can be regarded as a *correlation* between the states of the two electrons. This correlation of measurement outcomes occurs regardless of the distance of the entangled pair, so we can assume the two electrons have no interaction with each other. Einstein once famously derided the concept of entanglement as '*spooky* action at a distance', as he couldn't understand how correlation could arise without interaction [4].

Faster than Light

Someone may ask that this violates special relativity, since this thought experiment seems to allow superluminal transmission of information from Alice to Bob. However, this is not true, as explained below. The measurement outcome by Alice is entirely unpredictable (probability half of either state). Without Alice telling Bob, even if the state of electron A is already determined, it is still unknown and unpredictable to Bob. Therefore, although Bob knows that his measurement outcome is always the same as that by Alice, he can't predict it, since he can't predict what Alice got. Therefore, if Bob measures his electron, he has no way to tell whether Alice has made a measurement or not, unless Alice tells him. Moreover, if Bob does not measure his electron, he cannot

Only positive number to be directly between a square and a cube. Number of sporadic groups.



Entangled particles remain intimately and instantaneously linked throughout their existence.
Philipp Legner

predict his result unless Alice tells him her measurement outcome. In both cases, the process of Aice telling something to Bob will not be faster than light.

Entanglement is a fundamental feature of quantum mechanics, and a useful resource as well. Entanglement is a precedent to quantum non-locality, quantum teleportation and lots of other phenomenons or operations in quantum mechanics. Quantum non-locality is just the above phenomenon that measurement by Alice can instantaneously influence measurement by Bob.

Uncertainty

One of the other famous features of quantum mechanics is Heisenberg's uncertainty principle, which states that it is impossible to accurately measure both the position and momentum of a particle. Recently, researchers have uncovered surprising links between non-locality (arise from quantum entanglement) and the uncertainty principle [1]. They showed that the "amount" of non-locality is determined by the uncertainty principle. This is a dramatic breakthrough in our basic understanding of quantum mechanics.

Teleportation

Quantum teleportation, or entanglement-assisted teleportation, is a process by which the information of an electron (technically, a *qubit*) can be transmitted exactly from one location to another, without the electron being physically moved to the other location.

Suppose Alice has an electron in an unknown quantum state $|\phi\rangle$ (i.e. a and b are unknown) and she wants to pass this state to Bob. How can she do it? She has two obvious options:

- Physically carry this electron to Bob.
- Measure this electron to get information about a and b, then tell Bob so that he can recover the state.

Option 1 is highly undesirable, since quantum states are very fragile.

Option 2 cannot be implemented, since a measurement irreversibly changes the state of the electron. After the measurement, the state of the electron is fixed to $|0\rangle$ or $|1\rangle$, and you cannot recover $|\phi\rangle$ to make another measurement.

This seemingly impossible task can be done if Alice and Bob shares a pair of entangled electrons beforehand (can be the entangled state described previously). Alice just have to make a joint measurement of her two electrons (one of which has the unknown quantum state, the other one is part of the entangled pair) and tell Bob which state it is. Then Bob will perform some local operation to his electron according to what Alice gets. In this process, the original entanglement is destroyed and the electron which Bob possesses now has the unknown quantum state! Moreover, the two electrons that Alice has will now be entangled. The details are fairly technical. If you wish to know more, you are welcome to read the original seminal paper [3] or look up on Wikipedia [5]. Quantum teleportation has already been realised experimentally. Last year, a group of scientists from China managed to achieve quantum teleportation over a free space of 16km [2].

Quantum entanglement and teleportation form a central part of quantum information and quantum computation, which is an interdisciplinary field that draws from mathematics, physics and computer science, and one of the fastest growing area of research. If you are interested, you may consider joining it!

Is teleportation more than Science Fiction?
Philipp Legner, NASA

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How Electrons Spin

Natasha Kudryashova, Darwin College

It is almost impossible to construct a visual model for the microworld using analogies to macroscopic phenomena. The classical laws of physics governing the macroworld fail on the microscale, which is subject to quantum mechanical equations. For quantum particles, such as electrons, the two mutually exclusive classical concepts of wave and particle are complementary, resulting in a corpuscular-wave dualism of their properties. In addition, quantum particles, unlike macroscopic objects, only exist in discrete states characterised by a set of quantum numbers. For an elementary particle, spin is a part of its quantum state and is, therefore, a fundamental property.

According to the "macroscopic" view, spin should be associated with self-rotation of the particle on its axis. Though the quantum spin does represent an intrinsic angular moment it is not straightforward to imagine the corresponding rotation in view of the corpuscular-wave dualism. The configuration space for the quantum spin is very different from that of a spinning top, which visualizes classical self-rotation. Unlike classical spinning tops, elementary particles of a given kind always have the same spin, which is their fundamental property (like mass or charge), and cannot therefore "self-rotate" any faster or slower. Associated with self-rotation intrinsic degree of freedom is the spin direction (also referred to as spin), with the component of the spin along any direction taking only certain allowed values, i.e. being quantised. For an electron, projection of its spin of 1/2 (in "quantum" units of the reduced Plank constant $h/2\pi$), can acquire only two values: 1/2 and −1/2.

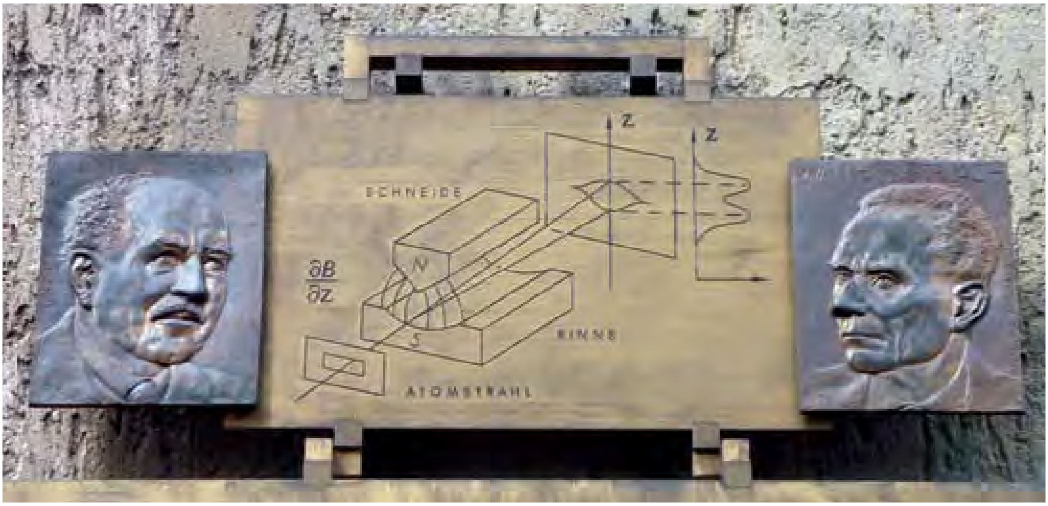
Furthermore, the position and momentum that uniquely define the angular momentum for a rotating macroscopic object cannot be specified simultaneously on the microscale by the Heisenberg principle, giving rise to probabilistic description of elementary particles. The statistics that the elementary particles obey are in turn fully determined by their quantum spin. For an electron, half-integer spin invokes the Pauli exclusion principle, which states no two electrons can have identical quantum numbers; this is a fundamental principle that underlies the structure of the periodic table.

What is Quantum Spin?

Historically, spin was introduced by Pauli as a "classically non-describable two-valuedness", to specify uniquely the quantum state of an electron [1]. The first experimental evidence that spin is real dates back to the beginning of the 20th century, when Stern and Gerlach observed unexpected splitting of spectral lines [2], which had not been interpreted properly until Pauli put forward the non-relativistic quantum mechanical theory of spin [3].

In 1928, Dirac [4] proposed his famous relativistic quantum mechanical equation for the electron where quantum spin appeared to be an essential part of the theory, as well as its inevitable consequence. Dirac showed that spin is essential to the relativistic wave equation, and that the proposed description of elementary particles with spin 1/2 is consistent with the principles of both quantum

Sum of the first four squares. Largest number such that all smaller numbers coprime to it are prime.



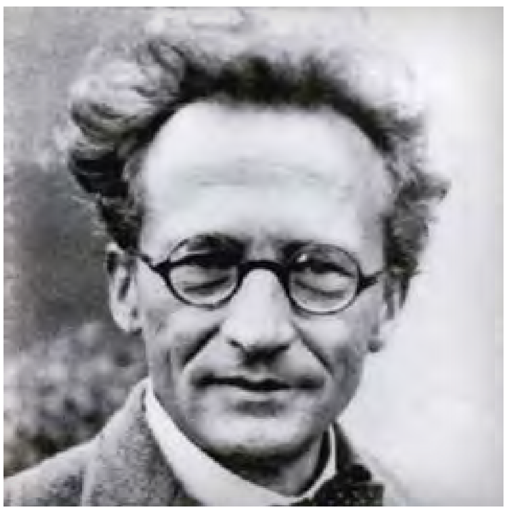
mechanics and special relativity. The Dirac equation can also approximately describe protons and neutrons, which are not elementary particles, but have spin of 1/2. Since then, great experimental evidence has proven Dirac's theory correct.

The most spectacular triumph of Dirac's theory was the discovery of positively charged particles with the mass of an electron (known as positrons). These were predicted by the Dirac equation, which requires existence of negative energy and density solutions.

It is well established that electron spin is neither the result of quantum mechanics, nor relativity [5]. Many have attempted to obtain spin equations as a classical limit of the Dirac equation, but different classical equations are derived according to the performance of the limits [6]. Numerous attempts to employ "macroscopic" ideas of rotation similar to a spinning top have also failed, as they are not consistent with the Dirac equations.

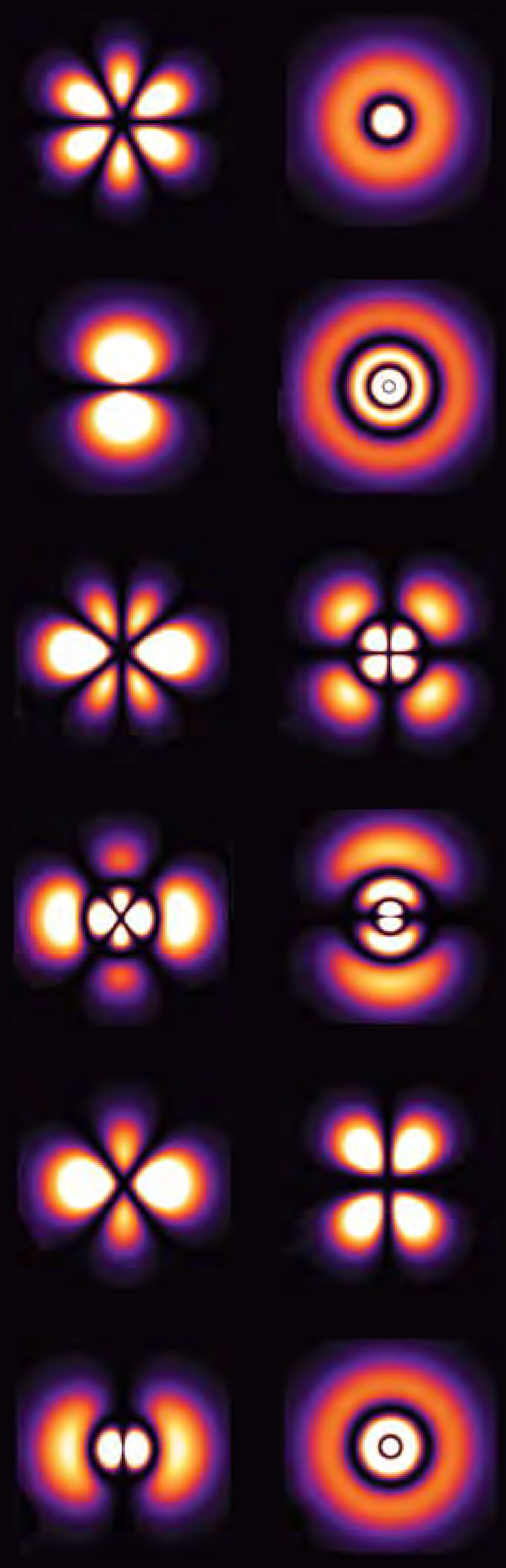
Trembling Electrons

A visual picture of the quantum spin, which is different from a classical spinning top, was proposed by Schrödinger [7], one of the fathers of the quantum theory. Having analysed the wave packet



▲ A plaque in Frankfurt, commemorating the Stern - Gerlach experiment. Stern was awarded the Nobel Prize in Physics in 1943. *Frank Behnsen*

▶ In 1940, Paul Dirac (top) wrote an article for Eureka explaining Quantum Mechanics. He shared the 1933 Nobel Prize with Erwin Schrödinger (bottom).



◀ Hydrogen wave functions of the electron in different quantum states.

▶ Schematic picture of a trembling motion.

solution to the relativistic Dirac equation for electrons in free space, Schrödinger showed that the instantaneous velocity of an electron equals the speed of light c, in contradiction to the fact that the observed speed of a massive particle, such as an electron, is always less than c. To resolve this apparent contradiction, Schrödinger proposed existence of rapid local "trembling" motion, with fluctuation at the speed of light of the position of an electron around the median with a circular frequency $2mc^2/h$ where m is the mass of electron, c is the speed of light and h the Plank constant. Then, the observed velocity of an electron is an average determined by measuring the electron's position over a small time interval [8]. This helical motion, named by Schrödinger the *Zitterbewegung*, provides an intuitive picture of the spin of electrons being its orbital angular momentum. Up to now, the Zitterbewegung is the only model of quantum spin that is consistent with the Dirac equation.

It is generally believed that the Zitterbewegung arises from interference of solutions to the Dirac equation with positive and negative energies and is a natural consequence of the corpuscular-wave dualism. By the Heisenberg principle, the position and the momentum of the electron cannot be specified simultaneously, giving rise to a Zitterbewegung. However, in order to get a visual picture, expectation values of corresponding quantum mechanical operators with wave packets are to be taken. If they are taken between the positive (or negative) energies only, the Zitterbewegung is well known to disappear.

The *Zitterbewegung*: Hypothesis or Reality?

Despite being predicted almost 80 years ago, the Zitterbewegung still remains very theoretical and has not been observed for the very system it was predicted for: a free relativistic electron. The reasons are that the system requires an extremely high frequency (1021 Hz), and an extremely high degree of localization in space (10–13 m). However, the Zitterbewegung is not entirely hypothetical. It is potentially observable in the non-relativistic limit of the Dirac equation using, for instance, two-dimensional carbon sheets known as graphene [10, 11].

In addition, for heavier particles under special conditions emulating relativistic conditions for a Dirac electron, the Zitterbewegung can be observed at much lower frequencies. Recently, the Zitterbewegung was simulated for photons in a two-dimensional photonic crystal [13] and finally observed for ions and ultracold atoms [12, 14], thus providing support for Schrödinger's picture of the quantum spin. However, whilst these experiments and numerical simulations with heavier particles do confirm the Zitterbewegung behaviour predicted mathematically, they do not provide a firm proof that electrons undergoes Zitterbewegung. Therefore, the ultimate test of the Zitterbewegung for a free electron is still to come.

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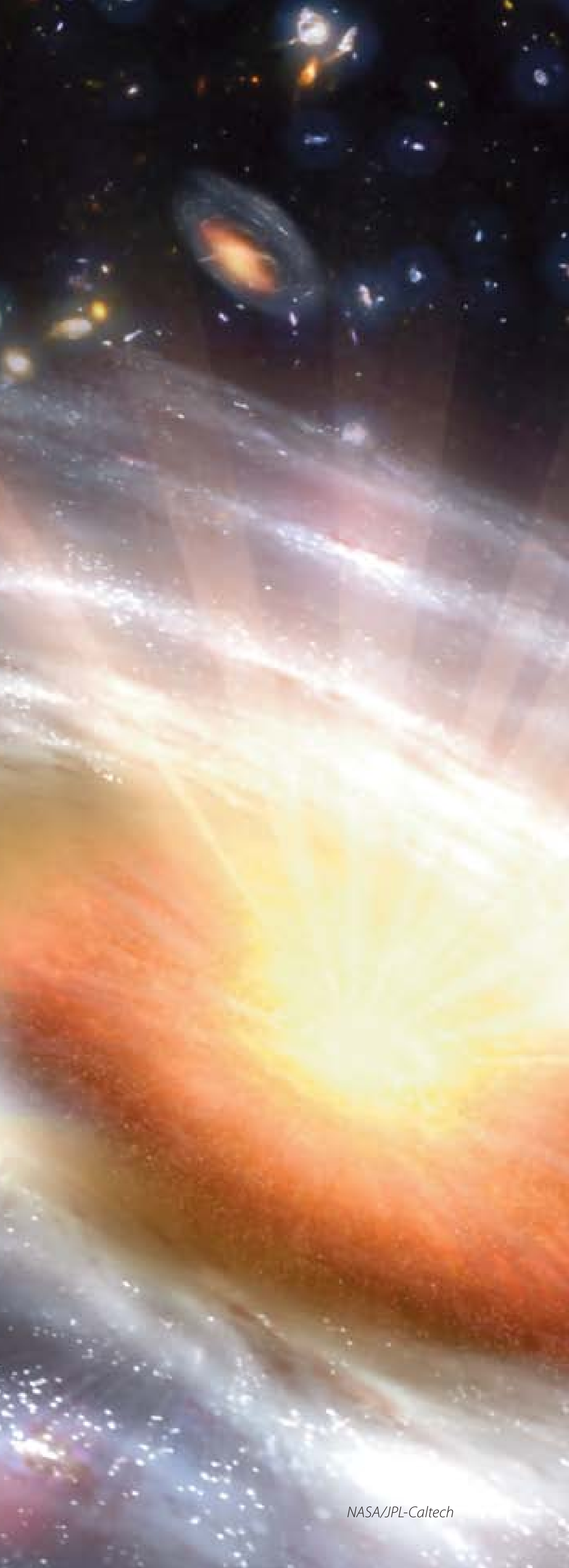

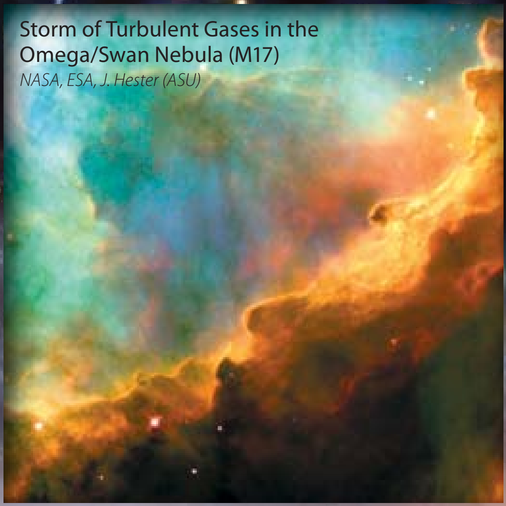

Storm of Turbulent Gases in the Omega/Swan Nebula (M17)
NASA, ESA, J. Hester (ASU)

X-ray/Visible/Infrared Image of M82
NASA, ESA, CXC, JPL-Caltech

The Antennae Galaxies/NGC 4038-4039
NASA, ESA, Hubble Heritage Team

Smallest number that has the same number of divisors as the previous and following number.

NASA/JPL-Caltech

News from the Beginning of the World

Sophie Dundovic, St John's College

Many academic disciplines spend a lot of time trying to predict the future and understand the past, yet the question remains: *why are we here now?* It has long been known that the physical constants which govern our existence fall within an extremely narrow band which would allow intelligent life to exist. Why this is has proven to be a topic of serious debate throughout the last century.

The *Anthropic Principle* states that we should not be alarmed by this lucky coincidence. Rather if the conditions were not right for conscious observers to exist, the conditions themselves would not be observed. Thus it is no coincidence that we find ourselves living at a time in space when it is possible for us to survive.

Relative to the age of our universe it was not long ago that we thought the world was flat. (Contrary to popular belief it was not Columbus who first refuted this claim; Hellenistic Astronomy established the sphericity of the earth as early as the 3rd century BC.) The early Astronomers and Mathematicians often developed radical ideas which have since been accepted as conventional wisdom. Thus whilst the theories at the forefront of cosmological research may seem counter intuitive today, they may well be tomorrow's reality. We have discovered that Earth is in fact not the centre of the Universe, that our galaxy is merely a small part of what we know to exist, but what about everything we don't know exists?

With advances in Quantum Field Theory accelerating all the time, Hugh Everett's *Many Worlds Interpretation* (MWI) is constantly provoking discussion. Of particular interest is its importance in the Multiverse Theory, advocated by Max Tegmark.

Is our Universe a Multiverse?

Assuming infinite space, which is much easier to envisage than finite space when you think about the boundary, as well as ergodic matter distribution, the ***Level I Multiverse*** predicts regions 'beyond our cosmic horizon'; that is regions of space which we are unable to observe. At present 42 billion light years is the farthest distance that we can observe, since that is the distance that light has been able to travel since the Big Bang. The Level I Multiverse is governed by the same laws of physics as the Universe we know, yet with different initial conditions.

The ***Level II Multiverse*** assumes that chaotic inflation occurred and asserts the existence of other post inflation bubbles such as our own universe which have the same fundamental physical equations but different initial conditions, physical constants, elementary particles and dimensionality as a result of chaotic inflation induced quantum fluctuations.

In a ***Level III Multiverse***, MWI comes into play. Essentially, every possible outcome of a random quantum event does occur. But if each decision a person makes is just a random quantum process undergone by the neurons in the brain then the Multiverse theory asserts that there should be an infinity of different histories all playing out at

The Galactic Centre Region
NASA, ESA, SSC, CXC, STScI

once, although perhaps in a different space-time. The further argument that if there are an infinite number of universes then there must be one at least which contains someone with your name your memories and your appearance seems weak. As Jesús Mosterín elegantly wrote,

> *"An infinity does not imply at all that any arrangement is present or repeated. [...] The assumption that all possible worlds are realised in an infinite universe is equivalent to the assertion that any infinite set of numbers contains all numbers (or at least all Gödel numbers of the [defining] sequences), which is obviously false."*

A Level III Multiverse is not falsifiable. Furthermore it places a dangerously strong importance on theories working on the observer. This could be criticised as venturing away from the rigour to which science seeks to adhere.

Finally the existence of a ***Level IV Multiverse***, which would consist of everything that exists, rests on two major assumptions, those being mathematical reality implying physical reality, and that the physical world is a mathematical structure. The existence of each level of Multiverses also depends upon the existence of the previous level.

Solving all Mysteries of Life

If Many Worlds is to be believed and were these assumptions to hold true then an "infinitely intelligent" mathematician should be able to compute equations to solve all of the mysteries of life and the universe. Not only that, but compute the futures of everyone and everything.

This would make Probability Theory, which is arguably one of the most useful tools available to mathematicians, superfluous. Yet the implausibility of the Level III Multiverse would void this. As every mathematician knows, if you start with wrong assumptions, you can prove that $2 + 2 = 3$.

It is admirable to have unfaltering faith in Mathematics, yet this is not a discipline built upon speculation and, whilst the fundamental belief that we can explain the physical world through a series of equations may be valid, radical interpretations of this idea can be misleading. The idea that any

Smallest non-trivial number which is both square and triangular. Sum of the cubes of the first three integers.

NASA, ESA, T. Megeath (University of Toledo), M. Robberto (STScI)

Star-Forming Region LH 95 in the Large Magellanic Cloud
NASA, ESA, Hubble Heritage Team

individual's actions could be predicted with total accuracy is not advocated by mathematicians seeking a TOE. It results again from trying to use Quantum Theory to observe the observer.

The Missing Link…

There is a missing link at the top; we have Quantum Field Theory to describe atoms, waves and particles, the smallest known matter and General Relativity to describe gravitation on a very large scale. Whilst they are incompatible with one another both theories are harmonious with the idea of a Multiverse.

It would be naive to think that simply because we cannot see beyond 42 billion light years that nothing exists past that point. In any case if there are infinitely many particles in space then space itself must be infinite. However space being infinite does not imply the existence of a Multiverse.

Perhaps we were a little ahead of ourselves in labelling the universe as we have done – with the prefix implying that it is alone, however it's definition is 'All existing matter and space considered as a whole; the cosmos.' Other variants include 'everything that exists.' Thus if the Multiverse were to be proven we would have to revise some definitions.

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Painting with Pixels

Martin Kleppmann, Corpus Christi Alumni
Alexander Hess, Universität Bonn

Alexander Hess uses numbers and mathematical structures to create digital art. His pictures are kaleidoscopic dreams filled with expressive colours and mysteriously intricate curves.

Throughout the ages, technical progress and the spirit of the age have been highly influential on the art of the period. Artists have always been inspired by new techniques and pigments, just as composers have been inspired by new instruments, or poets and writers by new developments in language. Hess's digital art is directly wired to the heartbeat of our time.

The imaging technique which Hess specially developed has only become feasible through state-of-the-art computing. He defines his visual creations through the complicated languages of computer programs: these programs are very carefully thought out, because once a picture has been generated, Hess will not modify it by hand. In other words, every picture is already fully contained in coded form in its program, and each picture requires a slightly different program. Hess paints in the abstract domain of computer instructions: numbers are his colours, mathematical operations are his brush strokes.

The pictures are not merely graphic toy applications of mathematics; that would be just as inappropriate as calling Damien Hirst's formaldehyde works "applied chemistry". Artistic expression has foremost importance for Hess, and mathematics is his medium of expression. "Some people consider my pictures to be somehow inferior, because they were painted by a computer program rather than a human hand", he says. "That is obviously nonsense. I don't mind whether an artist uses acrylic or oil paints, for instance. Every artist should use the media which work best for him. I am both artist and mathematician, and I cannot separate those identities."

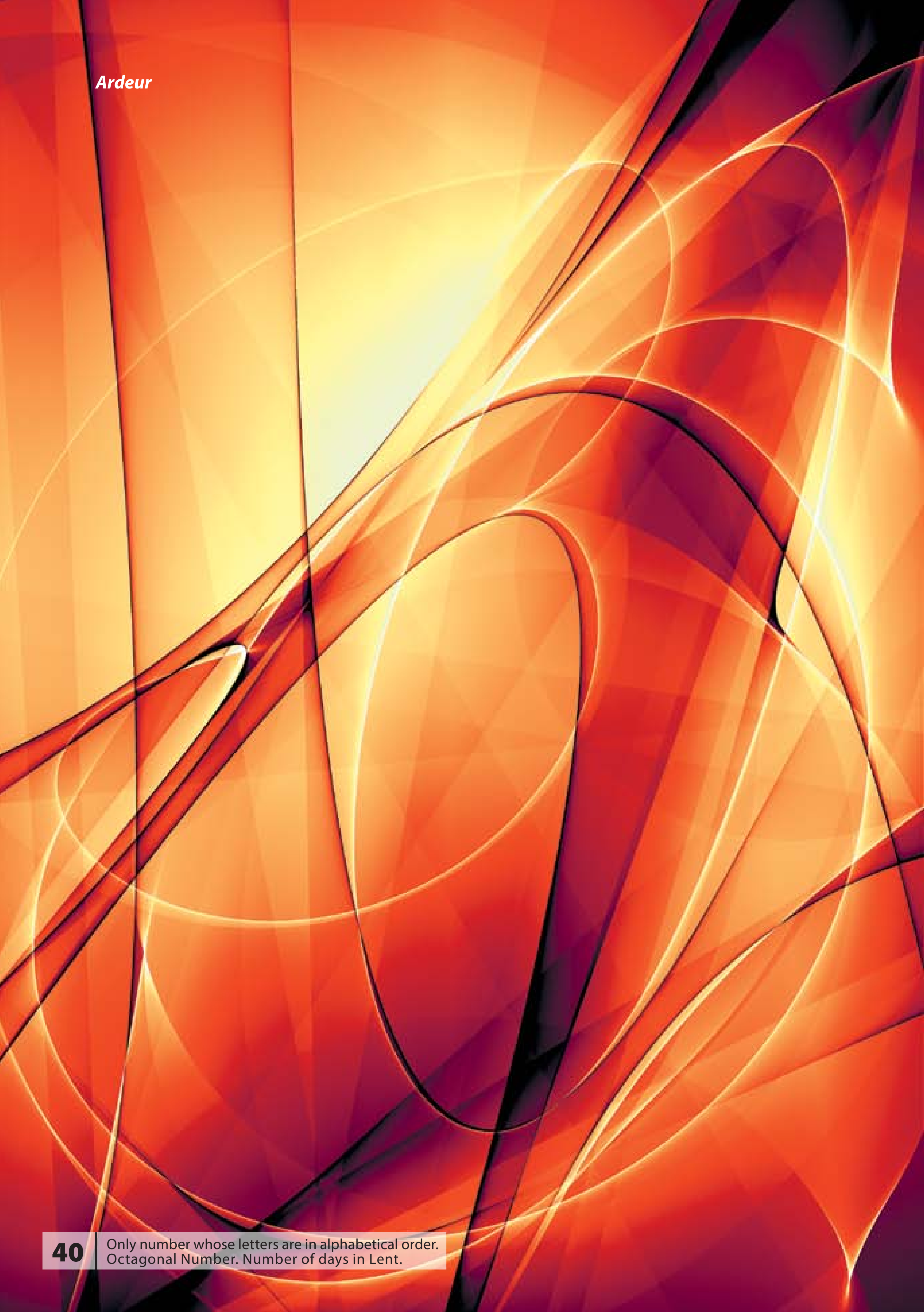
And Hess skilfully applies the special qualities of his digital means of expression. Today's huge computing facilities lend themselves to works of immense complexity. But rather than being dominated by this complexity and letting his pictures drift off into chaos, Hess chooses to restrain his computational daemons and let them work in the details. Many of his ideas are based on abstract, widely arcing curve motifs. The lasting fascination of his pictures often stems from the multilayered details: subtle shadings of colour; delicate and unexpected twists in the lines; curves which may exist or may only be imagined – and you are never quite sure.

As viewer, however, you do not need to know anything about these digital goings-on. You can just relax and regard Hess's visions like cloud formations from other worlds: sometimes you think you can see some concrete object or movement, and at other times they are simply a pleasing play of abstract forms and colours. Sometimes the title gives a hint at what Hess himself sees in an image; however that is not to say that a different viewer cannot gain completely different impressions from a look through the window into this artist's world.

Firework

Sum of consecutive primes (3 + 5 + 7 + 11 + 13). Number of steps in a Buchan novel and in a Hitchcock film.

Ardeur



40 Only number whose letters are in alphabetical order. Octagonal Number. Number of days in Lent.

La naissance de l'univers

The finished picture is composed of several abstract layers. In each of these layers simple geometric figures – such as lines, triangles, and circles – are drawn over each other millions of times. This is done by fixing one such object in each layer and deforming it continuously. Each of these figures has strong contours, but superimposing many similar pictures gives the impression of a very smooth pattern.

The colour of every single pixel in the final picture is calculated from these abstract layers. The method of assigning RGB-values from the layers can be defined in any way, and the methods vary between different pictures. The choice of function that assigns the colours has a profound effect on the final picture: the user decides which aspects of the picture are accentuated, e.g. by creating contrasts and choosing different colours.

How to teach Physics to Mathematicians

Zhuo Min 'Harold' Lim, St John's College

Mathematics and physics are close intellectual cousins. Historically, the development of both disciplines have been intimately tied together. For much of the late 18th century, and continuing through the 19th century to the early 20th century, the need to formulate physical theories and to perform calculations in physics was a major driving force in advances in such areas of mathematics as complex analysis, Fourier analysis, vector algebra and calculus, ordinary and partial differential equations, integral equations, and the calculus of variations. It may indeed be observed that many giants of this era (Euler, Gauss, Hamilton, Lagrange, Jacobi, Poincaré, Stokes, etc) were simultaneously great mathematicians and great physicists, having made significant contributions to both disciplines.

The emergence of the notion of mathematical rigour in the 19th century, along with the advent of the axiomatic method in the early 20th century, gave modern pure mathematics its present form. Mathematical writing is now expected to be precise; that is, definitions, results and proofs are supposed to be stated clearly and unambiguously (in practice, mathematical discourse is hardly ever that formal, to convey the ideas across more easily and to avoid boring the reader). There also emerged a trend towards abstraction and generalisation in mathematics: mathematicians turned to generalising older results for their own sake, rather than forging new mathematical tools for use in physics.

As observed by many people ([1] and [2]), this led to a kind of communication breakdown between the two camps. The situation has improved markedly in recent years, with mathematicians now collaborating with physicists working on the frontiers of fundamental theoretical physics (string theory, high energy physics, cosmology, etc). The need to formulate and solve problems in these areas has been a major driving force in the development of many branches of mathematics.

The Problem

Nevertheless, it may be observed that, in contrast to the situation a century ago, physics is no longer an important aspect of contemporary higher education in mathematics. Not many aspiring mathematicians (here, as in the rest of the essay, I use the term "mathematician" to refer only to pure mathematicians) are likely to take more than one or two courses in physics throughout the duration of his education. The pressure to learn the many different subjects that comprise the central core of mathematics, along with the technical and specialised fields needed for one's research, means that anything not considered essential (such as physics) will be given at best a perfunctory treatment.

Many more mathematicians will, however, end up working in subjects of great interest to theoretical physicists; for example, string theory and general relativity have made extensive use of modern geometry, while the representation theory of Lie groups has been very fruitful in illuminating quantum mechanics and field theory. Other mathematicians may go into other mathematical disciplines intimately tied to other sciences or industrial applications (nonlinear dynamics, inverse problems, free boundary value problems, percolation theory, etc). For these folks, a knowledge of physics, or at least a familiarity with physical thinking, which are hardly picked up during a typical undergraduate career in mathematics, would be an added bonus: physical considerations may help inspire new methods and results, and the ability to communicate with physicists (or scientists in general) could lead to many opportunities for collaborative work.

However, working mathematicians would no doubt bemoan the fact that physics is difficult to pick up. Both subjects have matured independently of each other over the decades, and one of the side-effects is that mathematicians find it extremely hard to adjust their thinking to that of a physicist. Also, mathematicians and physicists have different expectations in learning physics. These will be considered in greater detail below.

We are thus led to consider the following problem:

> *How can some basic physical intuition be imparted, and some knowledge of physics be taught to mathematicians, efficiently? Specifically, how should one present physics for mathematicians, or design physics courses tailored to the mathematicians' needs and tastes?*

Here, when using the term "mathematician" I am always referring to pure mathematicians who have completed a basic education in higher pure mathematics, and already possess a facility for abstract reasoning, but who have had very little training in physics. The point is to leverage this mathematical maturity so that the essentials of bare physics can be taught and learnt quickly. More importantly,

> *The goal is emphatically not to train mathematicians to become physicists, but to give them a "feel" of the practice of theoretical physics, so that they may communicate and collaborate better with physicists.*

The Programme of Applying Mathematics

In the practice of any scientific discipline in which mathematics plays a key role, the process of attacking a problem can usually be divided into three stages. They are detailed below, along with a few comments about how they might fit into teaching physics to mathematicians.

1. Creating a mathematical model

This is the stage in which a mathematical description of the phenomenon at hand is derived and written down. Very often, the resulting model is not an exact description of the phenomenon, but is simplified so that it is still a good approximation, but is mathematically tractable.

Mathematicians are likely to encounter trouble with this stage, especially after so many years of rigorous training in the formalised reasoning of pure mathematics. Modelling is very much more of an art rather than a science, and one has to use physical intuition to arrive at the equations and to make useful simplifications. Since there are no hard-and-fast rules to modelling, it is natural for one to find another's intuition difficult to follow or, worse, unconvincing.

2. Solving the problem mathematically

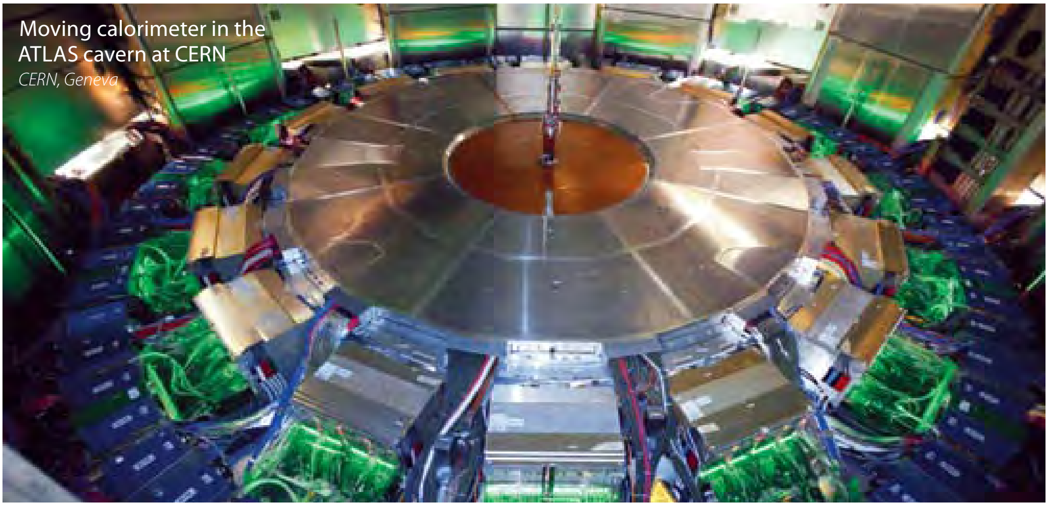
One then attempts to solve the model as a purely mathematical problem. Mathematicians will probably find this rather straightforward, once they learn the tricks of the trade.

3. Comparing the results to experimental data

One needs to conduct an experiment and measure quantities which can be compared with the results obtained. The model is useful if and only if its predictions (i.e. results obtained in the previous stage) are consistent with the data obtained in the experiments.

Naturally, mathematicians would have negligible training in this stage. However, this stage lies squarely in the domain of the physicist: it will take far too much time to train mathematicians in the techniques of experimentation, and, moreover, experimentation is obviously not the forté of mathematicians. One cannot expect mathematicians to make any useful contribution to the practice of this stage.

As an analogy, consider the teaching of the "mathematical methods of physics" to physics students. Much basic mathematics is omitted from such a curriculum, chief among them the techniques of rigorous proof. It is sufficient that physics students be able to use these "tools" to perform calculations, and it is unrealistic and unnecessary to expect them to be able also to engage in pure



Moving calorimeter in the ATLAS cavern at CERN
CERN, Geneva

mathematics. Similarly, and recalling our objectives outlined above, it is enough for mathematicians to develop a feel for physical thinking, and it is unrealistic and unnecessary to expect them to engage in experimental physics.

Consequently, experimentation is almost completely irrelevant to any course of physics designed for mathematicians.

Physics Courses are Unsuitable for Mathematicians

There may be many possible reasons why mathematicians find it hard to learn physics from physicists, physics books and physics courses. Here I will detail what I believe are the most important ones.

1. Differences in expectations

The fundamental reason is the difference in aims between the two subjects. Physics aims to give a mechanistic description of nature, and, to physicists, understanding is the ability to explain a physical phenomenon or process by more fundamental physical principles. Mathematics is viewed merely as a tool for expressing relations between observable quantities: the aim is solely to obtain a formula that can be used to compare theoretical predictions and experimental results. In particular, it does not matter at all if all the intermediate steps are un-rigorous (invoking assumptions without stating them) or incorrect (using a formula or relation that is known to be mathematically wrong) [4]. The answer is deemed to be valid as long as it accurately predicts experimental data (and, if the physicist in question is particularly sloppy, so is the method).

To a mathematician, there is something fundamentally distasteful about such a pragmatic use (abuse?) of mathematics. The mathematician strives instead for a logical understanding of the theory, and understanding is the ability to derive the result from the assumptions or hypothesis using indisputable rigorous argument. Unfortunately for them, physicists do not work in such a fashion, nor do they present their work in such a framework.

2. The lack of a clear delineation between the three stages of applied mathematics

This is likely to cause great difficulty for mathematicians in following physical reasoning. As mentioned above, modelling is conducted using heuristics such as physical intuition, instead of rigorous rules; as such, it may be the case that one might not consider another's modelling convincing.

Unfortunately, all too often the hapless student or reader, having been unconvinced by some stage of the modelling, would be stuck at this stage. This is a great shame because, even if one does not agree with the modelling, one can still proceed to solve the problem mathematically. There should be something left for the unconvinced mathematician to do. Parts which the mathematician should take on faith (i.e. the modelling stage), for example, as axioms, should be clearly separated from the parts which the mathematicians should work with (i.e. the mathematics stage).

Number of derangements of 5 items. Largest number of regions the plane can be divided into by 7 circles.

3. The lack of use of clear mathematical concepts

Physicists do not always use the most efficient language to convey physics. Mathematics has undergone tremendous progress since the early 18th century, yet the physicists' use of mathematics seems to be stuck in that era.

For example, in some (modern) treatments of general relativity, one still encounters the outdated concepts of "covariant tensors" and "contravariant tensors", defined as quantities that transform according to a certain rule, and make students or readers suffer through the practice of index shuffling. It would be much easier on the mathematicians to consider instead tensor products of the tangent and cotangent bundles; index shuffling is then a trivial consequence of this. Moreover, it becomes clear that "scalars" (such as the Ricci curvature) are smooth functions defined on the manifold of space-time and are trivially independent of the coordinate chart chosen, while "tensors" (such as the Riemann curvature tensor) transform correctly by taking suitable inner products with a basis of local vector fields.

As another example, I quote from Landau and Lifshitz's Mechanics [3],

> *It should be mentioned that this formulation of the principle of least action is not always valid for the entire path of the system, but only for any sufficiently short segment of the path. The integral (2.1) for the entire path must have an extremum, but not necessarily a minimum.*

This has the potential to cause great confusion: It seems to say, "whatever we have done so far isn't strictly true", but does not go into further detail why. Of course, it would have been far less confusing to use a little advanced mathematics, by stating that the path of the Lagrangian system is a geodesic on the phase space, and that locally (i.e. in an open neighbourhood of a point) geodesics are length-minimizing curves, but this is not true globally (for example, by considering great circles on the surface of a sphere).

Of course, in lectures designed for or books written for physicists, one will have to restrict the mathematical level so that the target audience could possibly understand them. However, this usually leads to a more fuzzy and less rigorous discourse. When presenting to mathematicians, the use of such mathematical concepts will illuminate the discussion considerably; not to use them would be inefficient.

Guidelines

Subject to the above discussion, I will now proceed to present my opinions on how physics could best be presented to mathematicians (in both lectures and books), so as to convey to them some notion of physical thinking (recall our goals above).

1. Clearly separate the modelling stage and mathematics stage. And ensure that, when solving the model mathematically, the use of mathematics is careful and rigorous: any results must follow from the governing equations of the model via a finite sequence of logically connected steps.
2. Use abstract mathematical concepts where they can illuminate the discussion.
3. Spend minimal time on experimental results. Seriously, mathematicians just aren't interested in that.

I would, in addition, envision such a course not so much as conveying knowledge of physics, as letting mathematicians try their hand at analysing some physically-motivated toy models. Such an analysis is necessarily mathematical.

It would, of course, be an added bonus if the mathematicians involved could be encouraged to try their hand at coming up with models. However, it should be remembered that modelling is emphatically not part of mathematics, and that mathematicians trained in the formalized reasoning of their discipline would definitely have to adjust their thinking. It is, as they say, difficult to teach an old dog new tricks.

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This

Penrose Tiling

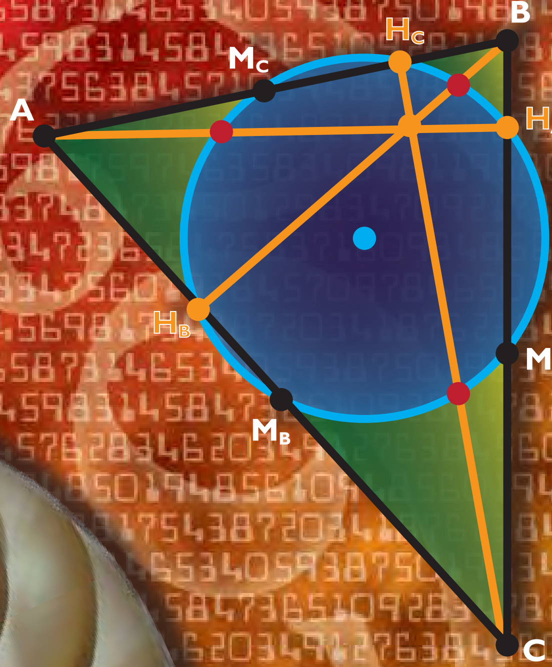
is an infinite aperiodic tiling of the plane, using only two kinds of shapes. It was discovered by Roger Penrose in the 1970s.

NUMBERS AND PATTERNS

The irrational number Pi is at the heart of the universe, defining the shape of a circle. Euler's

Nine Point Circle

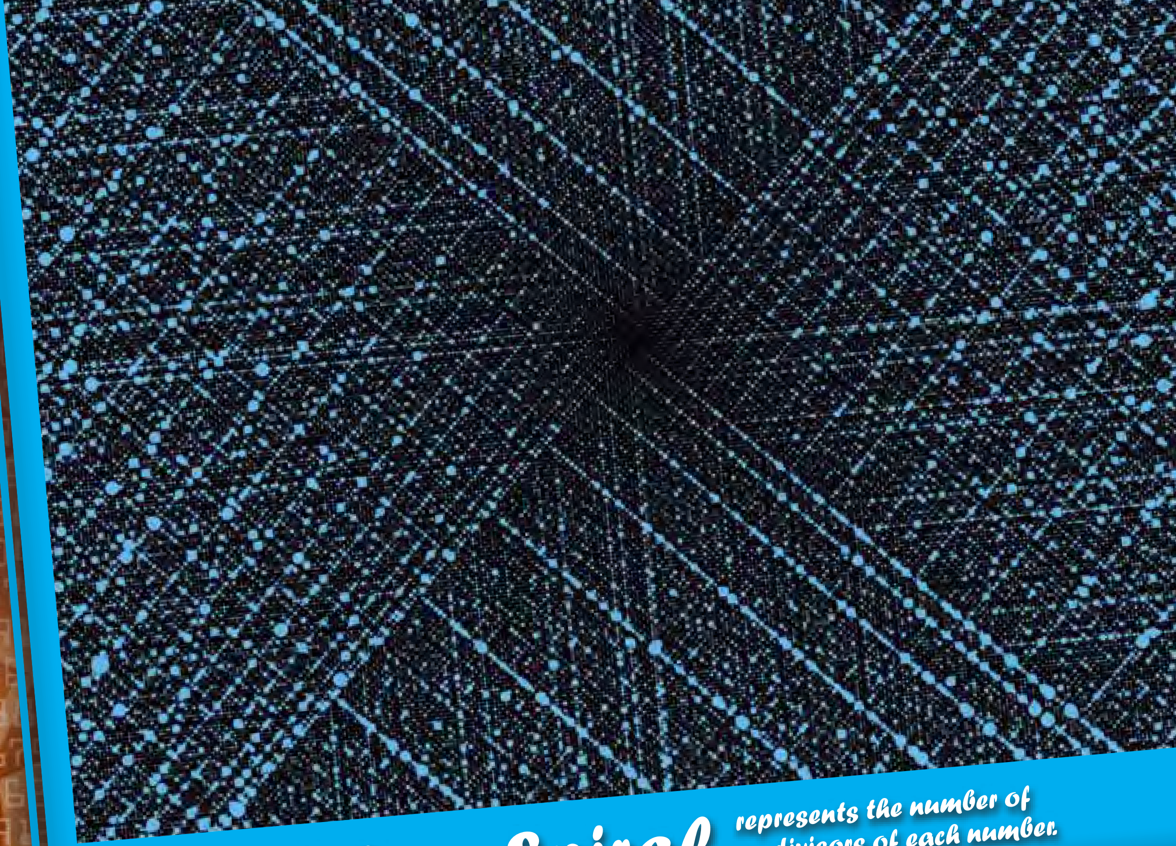
is a surprising example of how circles and triangles are connected.



While the occurence of the

Golden Ratio

in different parts of nature may be pure coincidence...



The size of the dots in the **Ulam Spiral** represents the number of divisors of each number.

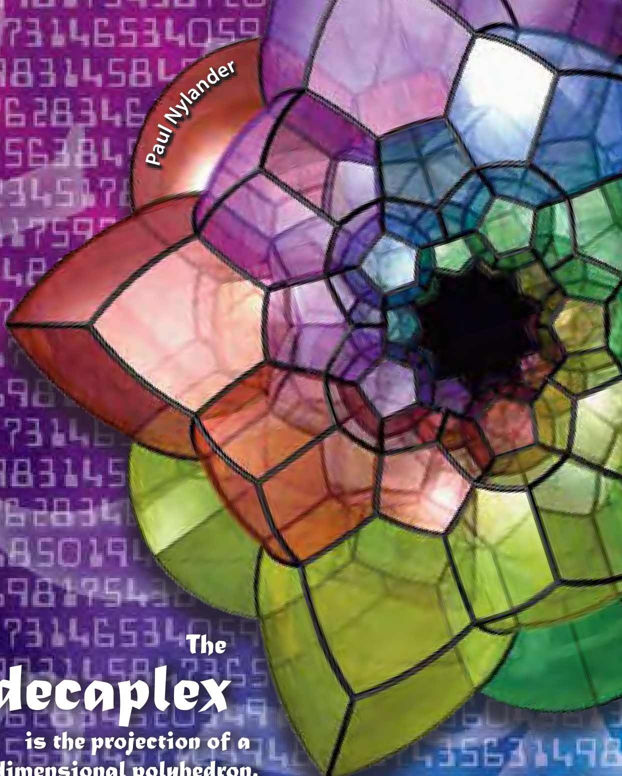
... the

Normal Distribution

is the underlying pattern of almost everything in nature: from human heights to beans in the game Quninqunx on the right.

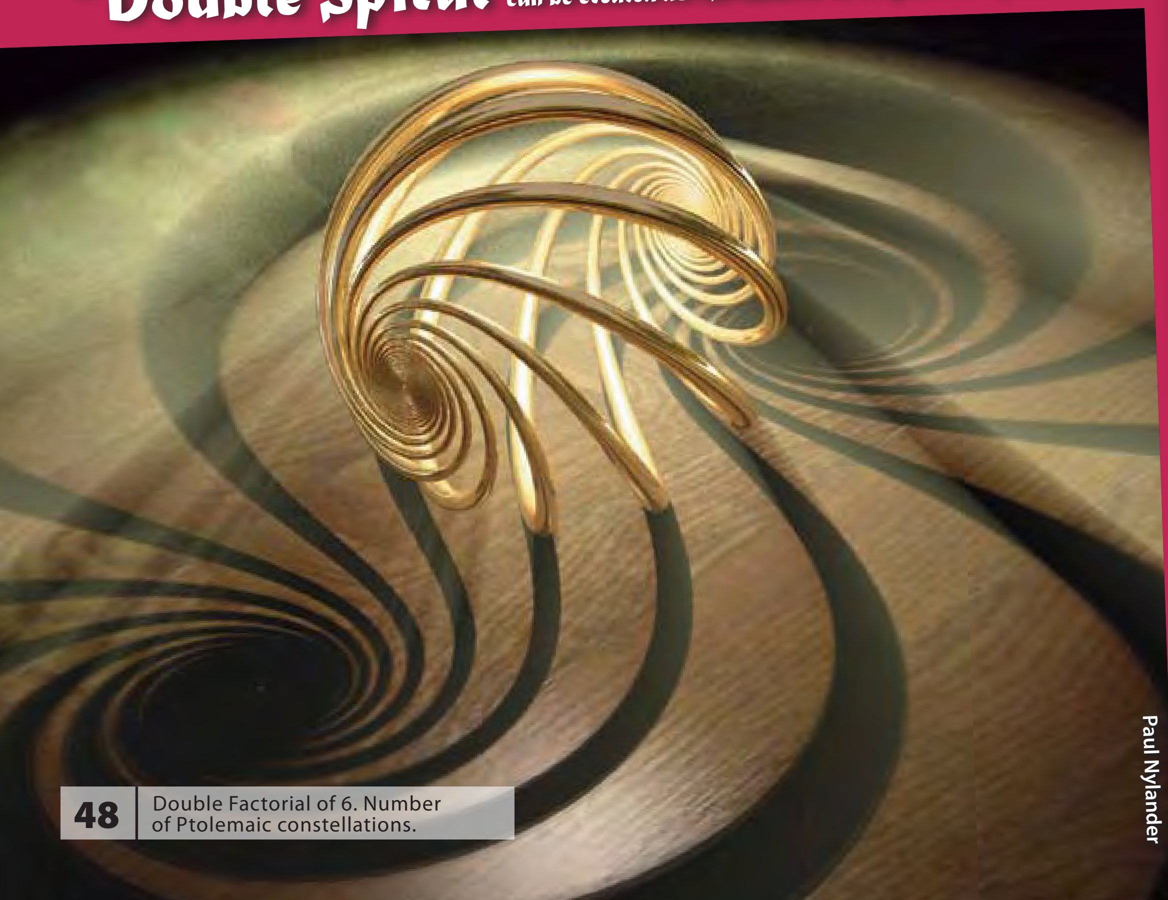
designed by Philipp Legner

The
Poincaré Disk
is one representation of the hyperbolic plane.



The
Dodecaplex
is the projection of a 4-dimensional polyhedron.

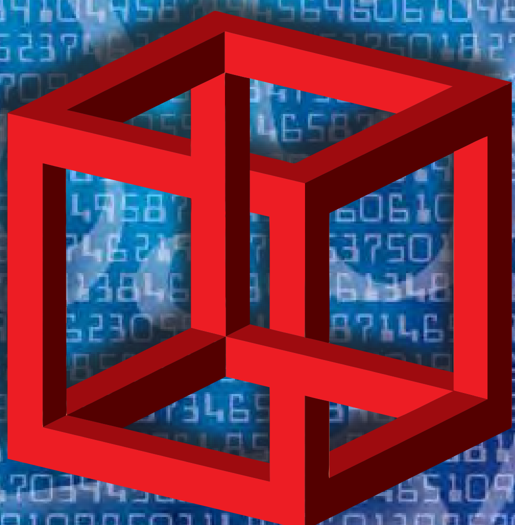
A **Double Spiral** can be created as the shadow of a spherical spiral.



Robert Scharain

Knots
are not only useful but give rise to an exciting mathematical theory.

SHAPES AND GRAPHS



The Necker Cube
is one of the impossible shapes M C Escher liked to use in his paintings.

Soap Bubbles
minimise the volume of a given surface area.

Antje van der Net

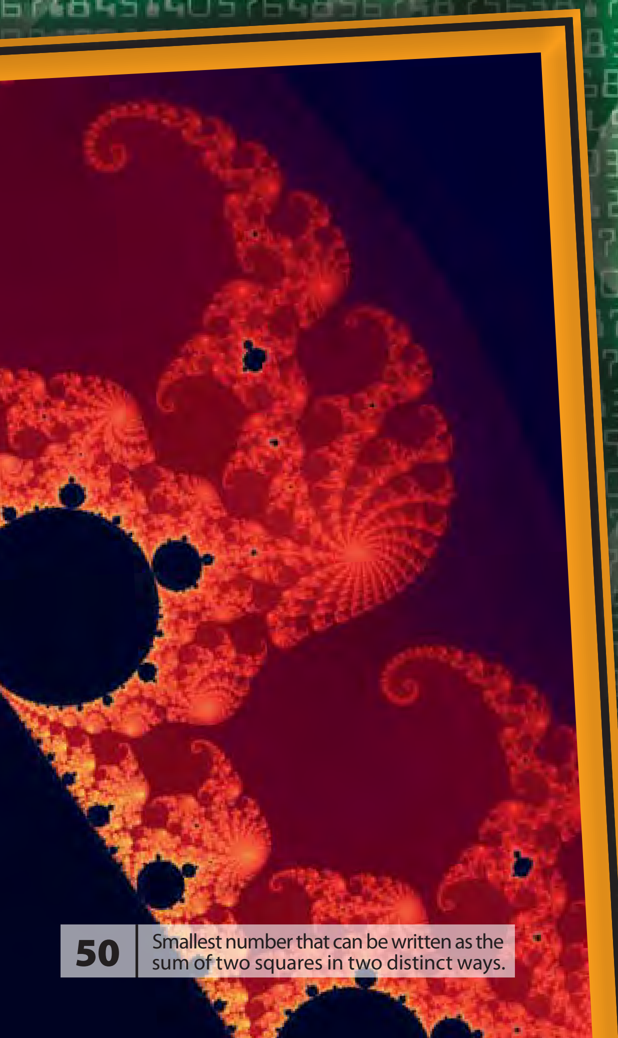
Boy's Surface

is only one of many interesting and beautiful mathematical surfaces, that can be created in elaborate computer applications.

The Menger Sponge

a 3-dimensional fractal

FUNCTIONS AND FRACTALS



Mandelbrot Set

is one of the most famous fractals of all times.

Stephen Wolfram's

Mathematica

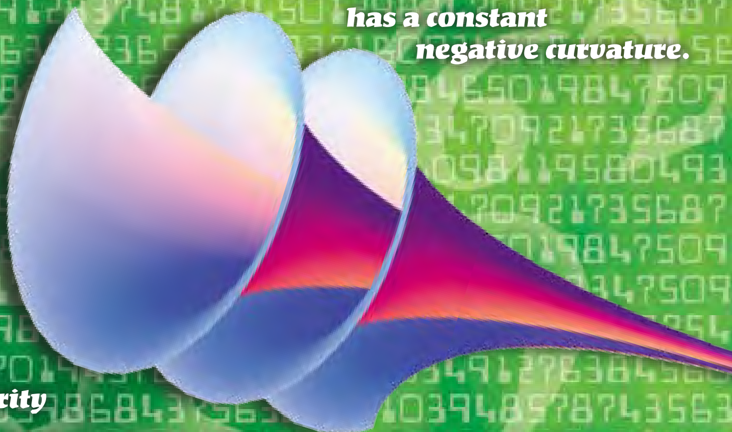
has changed the way we create and think abut mathematical graphics.

Waves

are one of the most important concepts in applied mathematics.

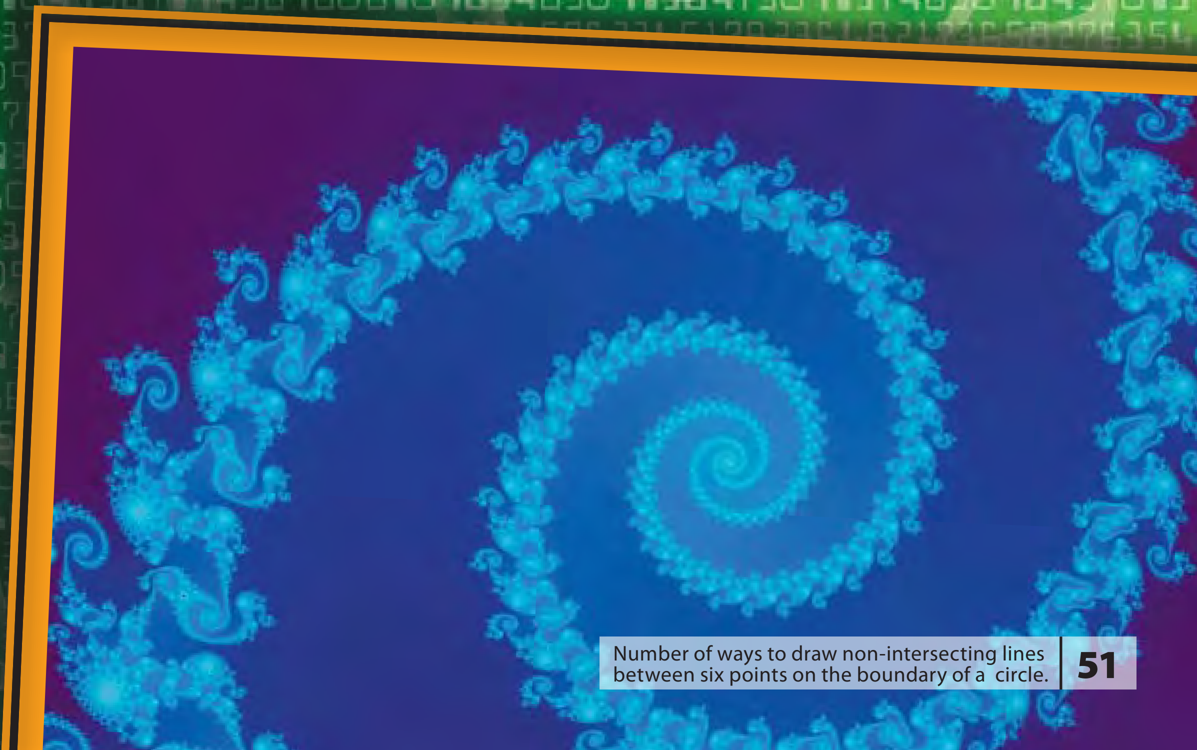
Dini's Surface

has a constant negative curvature.



Fractals

have found a fantastic popularity in recreational mathematics.



The Liar Paradox

Prof. Keith Devlin, Stanford University

The Liar Paradox, generally credited to the Greek philosopher Epimenides, asks us to consider the individual who stands up and says, "This assertion is false." Is this assertion true or false?

There are two possibilities: the speaker is either telling the truth or is lying. Let's look at each possibility in turn. To clarify the analysis, let *L* stand for the sentence uttered.

If the speaker is telling the truth, then the assertion must be true. According to what *L* says, that means that the speaker is uttering a falsehood. In other words, the assertion is a falsehood. But that can't be the case, since we started out by supposing that the speaker was telling the truth. So this case is contradictory.

Now let's look at the case where the speaker is lying. That means that the assertion is false. According to what *L* says, that means that the speaker must be telling the truth. Again we are in a contradictory situation.

It seems to be an inescapable paradox: if the speaker is telling the truth, then he or she is lying; if the speaker is lying, then he or she is telling the truth.

Until relatively recently, there was no known solution to this paradox. A key feature of the conundrum seems to be its self-referential nature. Certainly, the argument that leads to the paradox makes heavy use of the fact that the utterance of *L* refers to that very utterance. But self-reference alone cannot be the culprit. There is nothing inherently wrong with self-reference. It happens all the time, and often in polite company. People frequently talk about themselves. A group of people in conversation at a dinner party can talk about the dinner party, making such remarks as, "This is a very interesting conversation." And how about the self-referential sentence "This sentence has exactly six words." Attempts to resolve the paradox by analysing self-referential statements did not succeed.

Equally unsuccessful were approaches that concentrated on the notion of truth and falsity, another key ingredient of the paradox. One such attempt at a resolution was to suppose that there is a third possibility besides the assertion being true or false: it could be 'undetermined'. But the paradox arises again like the Phoenix from the ashes when someone stands up and says, "This assertion is false or undetermined."

Or perhaps the paradox depends on the particular combination of self-reference and a claim about truth and falsity. Certainly, there was no shortage of logicians who thought this was the case.

From America to Australia

Finally, in 1986, the American logicians Jon Barwise and John Etchemendy provided a solution. The cause of the difficulty, it turned out, was an unacknowledged, but critical, parameter – specifically, a context. Once you take proper account of the context in which the Liar sentence is uttered, there is no more a paradox than there is a genuine conflict between the American who thinks that

The background is in front of the horse behind the ▶ trees: a paradoxical work by Surrealist René Matisse.
ADAGP Paris and DACS London

June is a summer month and the Australian who thinks June is a winter month. Both individuals are correct in the context in which they hold their respective beliefs, America on the one hand, Australia on the other.

Before I present Barwise and Etchemendy's solution, I should point out that the Liar Paradox is not about sentences but utterances (of sentences). On its own, a sentence is neither true nor false. Truth and falsity arise only when someone utters the sentence (or writes it, or otherwise endows it with meaning). Thus, presentations of the paradox that focus on the sentence "This sentence is false" confuse a string of words with the meaning those words convey when someone utters them. That is why I used the word *assertion* when I stated the paradox.

Everyone knows implicitly that the circumstances in which a sentence is uttered affect the meaning in a fundamental way. For instance, when uttered by a person with the appropriate authority vested by society, the sentence "I now declare you man and wife" has significant ramifications for the two recipients. When the same sentence is spoken by an actor in a movie, those ramifications do not follow.

There is an entire branch of mathematics that investigates the way context affects meaning (and hence truth) in mathematics, called Model Theory. The notation model theorists use to indicate that a sentence σ is true when interpreted in the context M is

$$M \vDash \sigma.$$

For example, if $\mathbb{Z}$ denotes the integers, then

$$\mathbb{Z} \vDash (\forall x)(\forall y)[x \times y = y \times x],$$

but if M denotes the domain of all 3 × 3 integer matrices then

$$M \vDash \neg(\forall x)(\forall y)[x \times y = y \times x].$$

The formula is the same in both cases, $(\forall x)(\forall y)[x \times y = y \times x]$, the commutative law for multiplication. Whether it is true or false depends on what it is being applied to (i.e. in what mathematical structure it is being interpreted). Absent an appropriate context, the commutative law makes no claim, and hence is neither true nor false.

When you apply these considerations to the Liar Paradox, it simply melts away. Here is the argument.

Person a stands up and says, "This assertion is false". As before, let L denote the sentence uttered. The first question to ask is what exactly the speaker refers to by that phrase "This assertion." It cannot be the sentence L itself. As we noted above, sentences are just strings of symbols, and a string of symbols is neither true nor false, it's just a string of symbols. Rather what the speaker is referring to must be the assertion (or claim) being made by uttering the sentence. Let's call that assertion p (for *proposition*). In other words, a's utterance of the phrase "This assertion" refers to the claim p.

It follows that, in uttering the sentence "This assertion is false", a is making the claim 'p is false'. But we already used p itself to denote the claim made by a. Hence p and 'p is false' must be one and the same. I'll write this as an equation and give it a number to refer to later:

$$p = [p \text{ is false}]. \qquad (1)$$

Now, the claim p made by a's utterance concerns the truth of p. But, as we have observed already, if we want to be able to decide whether a particular claim is true or false, we need to be careful about the context in which the claim is made. In other words, in making the assertion – which is about that very assertion – a must be making implicit reference to the context in which the assertion is made. Let c denote that context.

Thus, a's utterance of the phrase "This assertion" refers to the claim that p is true in the context c. Using the notation of model theory, this can be written as $c \vDash p$. In other words, p must *be* the same as $c \vDash p$, since both are what a refers to by uttering the phrase "This assertion". So we have established a second equation:

$$p = [c \vDash p]. \tag{2}$$

Having sorted out what a is talking about, it's time to see whether a's assertion is true or false.

Suppose first that a's assertion is true. In other words, p is true. Using formula (2), we can express this as

$$c \vDash p. \tag{3}$$

By formula (1), we can replace p in formula (3) by [p is false] to obtain:

$$c \vDash [p \text{ is false}]. \tag{4}$$

Now we have a contradiction: formula (3) tells us that p is true in the context c and formula (4) tells us that p is false in the same context c. Notice that there is no question of an America–Australia type context difference here to explain the conflict. The context is the same in both cases, namely c. The contradiction is inescapable, just as if we suddenly discovered it was simultaneously summer and winter in San Francisco. (Actually, anyone who has visited San Francisco in July will know that this often seems to be the case, but that is a reflection of the strange summer weather there, not an issue of logic.) The only possible way out of this dilemma is that a's claim cannot be true, since that is the supposition that got us to the contradiction.

So much for the case when we assume that a's claim is true. Now let's look at the case where a's claim is false. In other words,

$$p \text{ is false.}$$

But wait a minute. What is the context for this statement? This question did not arise in the previous case, since we knew the context for p; it was c. But nothing we know provides a context for the statement 'p is false'.

You might feel that c is itself the appropriate context. After all, c is the context in which a makes the assertion, and to which the assertion implicitly refers. Fair enough, let's see what happens if we do make this assumption. Then

$$c \vDash [p \text{ is false}].$$

By formula (1), this can be rewritten as

$$c \vDash p.$$

And now we are in the same contradictory situation as we were in the previous case. On that occasion the conclusion was that a's claim cannot be true. But this time the conclusion is different: namely that c cannot be the appropriate context. Just as the knowledge that if a person in country X says truthfully that June is a winter month leads us to conclude that country X is not America, so too on this occasion, if a's claim is false, then we can conclude that the context for that claim being false cannot be c.

You have to be a bit careful with the above comparison with the America–Australia example. c is indeed the context for a's utterance. What is at issue is what is the context for making the new statement that a's original utterance is false. What the above argument shows is that *that* context, whatever it is, cannot be c.

So, when proper attention is paid to context, the Liar Paradox ceases to be a paradox. In saying "This assertion is false", the individual a is making a claim that refers (implicitly) to a particular context, c, the context in which the sentence is uttered. If the claim is true, then it is true in the context c, and that leads to a contradiction. So the claim must be false. But the context for making the observation that the claim is false cannot be c, since if it were, then that too leads to a contradiction.

In other words, what was previously regarded as a paradox has turned into a discovery, or *theorem*, about contexts. A person a who stands up and says (in context c), "This assertion is false", is making a false statement. However, that fact that the statement is false cannot be asserted in the same context c. Admittedly this is a fairly odd conclusion. Then again, the Liar sentence is a pretty odd thing for anyone to say.

Number of coloured squares on a Rubik's cube.
Perfect round on a par 72 golf course.

There is a particular irony about the way the Liar Paradox was finally put to rest. The puzzle was first formulated by a logician in ancient Greece, at the time when the Greeks were starting to develop a theory of reasoning and truth (i.e. the theory we now call Logic) that was independent of context. And yet more than two thousand years later we are able to recognize that Epimenides' argument is really about the crucial role played by context in discussing reasoning and truth. In short, a proper analysis of both communication or reasoning cannot be carried out without taking account of context. The Liar Paradox is not a paradox at all, rather the nonsensical outcome of bad mathematical modeling.

Notes

1. Bertrand Russell's famous set-theoretical paradox about the set of all sets that do not contain themselves was likewise resolved by identifying an unacknowledged parameter. Russell considered the set $R = \{x \mid x \in x\}$ and reached a contradiction by asking whether $R \in R$. Though Russell's Paradox destroyed Gottlob Frege's logical magnum opus before it was published, it did not take long before the resolution was found, by way of a proper axiomatisation of set theory. The crucial set-formation axiom says that, given any set s and any property P of sets, it is possible to form the set $\{x \in s \mid P(x)\}$. Frege's formulation omitted reference to the crucial parameter s, and thereby opened the door to Russell's devastating example.

2. Barwise and Etchemendy's solution to the Liar Paradox was first described in [1]. For examples of the kinds of results that can be obtained when the methods of model theory are applied to real-world situations, as opposed to mathematical structures (for which the methods were first developed), see [2].

References, Further Reading

1. J. Barwise, J. Etchemendy, *The Liar: An Essay on Truth and Circularity*, Oxford University Press (1987)
2. K. Devlin, *Logic and Information*, Cambridge University Press (1991)

About the Author

Dr. Keith Devlin was born and grew up in Yorkshire, earned his BSc in Mathematics at Kings College London, and his PhD in Mathematics at the University of Bristol. He has lived in the USA since 1987, and has been at Stanford University since 2001. He is a co-founder and Executive Director of the Stanford H-STAR institute, a co-founder of the university's Media X research network, and a Senior Researcher at the CSLI center. He is a World Economic Forum Fellow and a Fellow of the American Association for the Advancement of Science. His current research is focused on the use of different media to teach and communicate mathematics to diverse audiences. He also works on the design of information/reasoning systems for intelligence analysis. Other research interests include: theory of information, models of reasoning, applications of mathematical techniques in the study of communication, and mathematical cognition. He has written 31 books and over 80 published research articles. Recipient of the Pythagoras Prize, the Peano Prize, the Carl Sagan Award, and the Joint Policy Board for Mathematics Communications Award. In 2003, he was recognized by the California State Assembly for his "innovative work and longtime service in the field of mathematics and its relation to logic and linguistics." He appears regularly on National Public Radio, where he is known to millions of listeners as "the Math Guy".

When Logic Meets Geometry

Tom Avery, St John's College

Many important advances in mathematics are made when connections are found between seemingly unrelated topics. In recent years, one such connection has been made between the fields of Model Theory and Algebraic Geometry, using the concept of a *Zariski structure*. A Zariski structure is a model theoretic structure – a set with a collection of relations between its elements. We also require some topological information about the structure. Under certain conditions, it can be shown that such a structure is very closely related to an algebraic variety – the main objects of study of Algebraic Geometry.

"I began to feel distinctly unhappy about the rigor of the original proofs [of the theory of surfaces], without losing in the least my admiration for the imaginative geometric spirit that permeated these proofs. I became convinced that the whole structure must be done over again by purely algebraic methods."

Oscar Zariski

In Algebraic Geometry, we study the geometric properties of sets of solutions of polynomial equations over a field. if $f_1, \ldots, f_m$ are polynomials in n variables with coefficients in a field K, sets of the form $\{(x_1,\ldots,x_n) \in K^n : f_i(x_1,\ldots,x_n) = 0 \text{ for } i = 1,\ldots,m\}$ are called algebraic. The algebraic sets in K^n are the closed sets of a topology on K^n, called the Zariski topology. Sets which are the intersection of an open and a closed set are called constructible, because they are precisely the sets which can be constructed from algebraic sets using Boolean operations. An algebraic set that is irreducible, that is that cannot be written as the union of two proper closed subsets, is called an algebraic variety. We can also define a notion of dimension for constructible sets. If $S \subseteq M^k$ is constructible and irreducible, we define $\dim(S)$ to be the maximal value of n such that there is a chain of proper closed irreducible subsets of $S = S_n \supsetneq S_{n-1} \supsetneq \cdots \supsetneq S_1 \supsetneq S_0$. The fact that this is well defined (i.e. that there is such a maximal n, and it is finite) is a consequence of the fact that the polynomial ring $K[X_1,\ldots,X_k]$ is Noetherian.

Model Theory

Model Theory is the abstract study of mathematical structures. We start with a set M, called a structure, with a collection $\mathcal{C}$ of subsets of M^n, which we think of as relations between elements of M, for example, if $P \in \mathcal{C}$ and $(x_1,\ldots,x_n) \in P$ then

$P(x_1,\ldots,x_m) \wedge Q(y_1,\ldots,y_n)$	$(P \times M^n) \cap (M^m \times Q)$
$P(x_1,\ldots,x_m) \vee Q(x_1,\ldots,y_n)$	$(P \times M^n) \cup (M^m \times Q)$
$\neg P(x_1,\ldots,x_m)$	$M^m \setminus P$
$\forall x_1 P(x_1,\ldots x_m)$	$\{(x_2,\ldots,x_m) : (x_1,\ldots,x_m) \in P \text{ for all } x_1 \in M\}$
$\exists x_1 P(x_1,\ldots x_m)$	$\{(x_2,\ldots,x_m) : (x_1,\ldots,x_m) \in P \text{ for some } x_1 \in M\}$

Oscar Zariski (1899–1986)

we say that $x_1,\ldots,x_n$ satisfy the relation P, and write this as $P(x_1,\ldots,x_n)$. The relations in $\mathcal{C}$ make up the "atomic formulae" of M. We define formulae inductively starting from the atomic formulae using logical operators. If $P \subseteq M^m$ and $Q \subseteq M^n$ are formulae, then the table on the previous page shows how we form new formulae from them; the first column gives the formula, and the seconds gives the set of elements satisfying it.

The set L of all formulae is called the *language* of M, and the corresponding subsets of M^n (for each n) are called the *definable sets*. Model theory is an incredibly powerful tool for describing mathematical structures; there are very few mathematical objects that cannot be thought of as a model-theoretic structure.

We call a structure M a topological structure if the subsets of M^n defined by atomic formulae are the closed sets of a topology on M^n, for each n, and the topology is well behaved under certain set theoretic operations. In particular, we require that

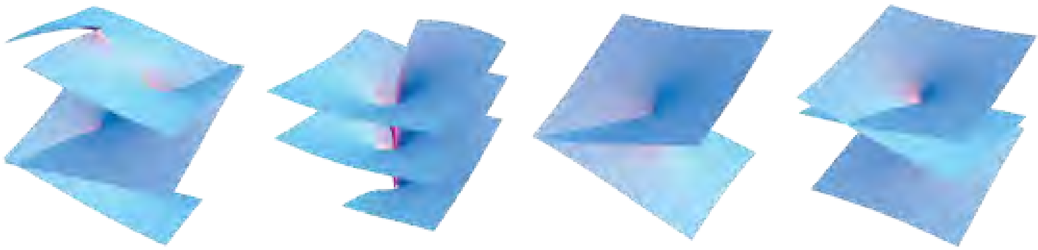
1. $\{(x,x) : x \in M\} \subseteq M^2$ is closed,
2. Every singleton set is closed,
3. The Cartesian product of two closed sets is closed,
4. Permuting the coordinates of a closed set gives a closed set,
5. If $a \in M^m$ and $S \subseteq M^{m+n}$ is closed, then $\{x \in M^n : (a,x) \in S\}$ is closed.

Constructible subsets of a topological structure are defined in the same way as for a field – any set which is the intersection of an open and a closed set. We say a topological structure has "good dimension notion" if for every non-empty definable set S there is a non-negative integer $\dim(S)$, satisfying the following:

1. The dimension of a singleton is 0,
2. $\dim(S_1 \cup S_2) = \max\big(\dim(S_1), \dim(S_2)\big)$,
3. For constructible irreducible S, if $S_1 \subsetneq S$ is closed, then $\dim(S_1) < \dim(S)$,
4. If $S \subseteq M^n$ is constructible and irreducible, and $\mathrm{pr} : M^n \to M^m$ is a projection map (i.e. $\mathrm{pr}(x_1,\ldots,x_n) = (x_{i_1},\ldots,x_{i_m})$, where $i_1,\ldots,i_m \in \{1,\ldots,n\}$), then
$$\dim(S) = \dim(\mathrm{pr}(S)) + \min_{a\in\mathrm{pr}(S)} (\dim(\mathrm{pr}^{-1}\{a\} \cap S)),$$
5. For $S \subseteq M^n$ constructible and irreducible, and $\mathrm{pr} : M^n \to M^m$ a projection map, there exists $V \subseteq \mathrm{pr}(S)$ open in $\mathrm{pr}(S)$ (in the subspace topology), with
$$\min_{a\in\mathrm{pr}(S)} \dim(\mathrm{pr}^{-1}\{a\} \cap S) = \dim(\mathrm{pr}^{-1}\{v\} \cap S)$$
for every $v \in V$.

A topological structure is Noetherian if for every n, if we have a descending chain of closed subsets in M^n, $S_1 \supseteq S_2 \supseteq \cdots$, then there is some i such that $S_i = S_j$ for every $j \geq i$. Finally, a Zariski structure is a Noetherian topological structure with good dimension notion, such that for any closed irreducible $S \subseteq M^n$ and projection map $\mathrm{pr} : M^n \to M^m$, there is a proper closed subset $F \subset \overline{\mathrm{pr}(S)}$ such that $\overline{\mathrm{pr}(S)} \setminus F \subseteq \mathrm{pr}(S)$ (here $\overline{\mathrm{pr}(S)}$ denotes the closure of $\mathrm{pr}(S)$).

Some of the above definitions might seem somewhat arbitrary, but in fact they have been chosen carefully so that Zariski structures behave very similarly to algebraic varieties. An algebraic variety is a model theoretic structure with atomic formulae defined by finite sets of polynomials over the field K. In fact, this makes the variety a topological structure, since as noted above the algebraic subsets form the closed sets of a topology, and the other requirements for a topological structure are quite straightforward to check. If we define dimension in an algebraic variety as above, it is not too difficult to verify that an algebraic variety is in fact a Zariski structure. The main result in the theory of Zariski structures so far is the classification theorem, which establishes a partial converse to this.



Compact Riemann surfaces are examples of Zariski ▲ structures. These Riemann surfaces correspond to ArcSin(z), Log(z), $z^{1/2}$ and $z^{1/3}$ (left to right).

Classification Theorem

Given a one-dimensional essentially uncountable, ample, pre-smooth[1] Zariski structure C, there is a field K, an algebraic curve X over K and a finite-to-one surjective map

$$f : C \to X$$

such that the image of a definable set in C is constructible, and the pre-image of a constructible set in X is definable.

Notice that the map f is only *finite*-to-one, not *one*-to-one (injective). If it were, then the Zariski structure C and algebraic curve X would be exactly isomorphic as Zariski structures. In fact there are examples where the map is not one-to-one, and so the Zariski structure is not an algebraic curve, only a finite cover of one.

The proof of the classification theorem is too long to give here in full, but we can at least describe some of its interesting features. The most surprising part of the result is the construction of a field – we begin with a purely logical structure (with a little bit of topology thrown in) and end up with a field, an algebraic object definable within the structure. Once the field has been found it is relatively straightforward to construct the curve X.

To construct the field, we must first consider an elementary extension $^*C$ of C – a much larger model theoretic structure which contains C as a substructure – and a map $\pi : {}^*C \to C$ called a specialisation. We think of $^*C$ as containing many extra points squeezed in between the points of C, and π takes each of these to a point of C which is "infinitesimally close" to it. The pre-image of a point in C under π is called its infinitesimal neighbourhood. We can use such an extension to formulate "non-standard analysis" in our Zariski structure.

1 essential uncountability, ampleness and pre-smoothness are technical conditions on the Zariski structure, that serve to rule out certain degenerate cases. For the exact definitions, see Boris Zilber's book, "Zariski Geometries"

The next stage is to consider a family of curves (i.e. one-dimensional closed sets) in C^2 through a point (a,b). Non-standard analysis shows that each of these curves is in fact the graph of a bijection when we restrict it to the infinitesimal neighbourhood of (a,b). We can invert these functions and compose them, and in doing so define a new family of curves in C^2 through (a,a). Composing curves from this new family again gives curves through (a,a). There is an equivalence relation of tangency between curves through a particular point, and the composition of curves preserves this relation, so there is a binary operation on the set of tangency classes of curves through (a,a). In fact this operation defines an abelian group, G, which is definable within our Zariski structure.

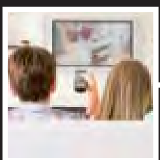
This process can be repeated, except using G instead of our original structure C. We consider a family of curves through $(0,0)$ (where 0 is the identity of the group). There are now two operations on the family of curves: pointwise addition of the local functions, and composition as before. Both of these preserve tangency, and writing composition as multiplication, they define a field structure on the set of tangency classes of curves through $(0,0)$.

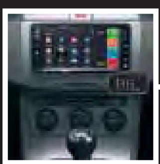
Further Reading

There are many examples of Zariski structures in mathematics, the most important being algebraic varieties and compact Riemann surfaces. The study of these structures is ongoing, and the idea of an analytic Zariski structure, in which the condition that the topology be Noetherian is dropped, has received much interest in recent years. If you would like to find out more, I suggest Boris Zilber's book on the subject, "Zariski Geometries".










Graphs, Groups and Topology

Anja Komatar, Queens' College

As undergraduates we meet group theory, graph theory and topology as distinct courses, but there are many ways in which these areas interact. The following five examples are particularly interesting.

Conjugacy in S_n

Consider a bijection f from the set $\{1, 2, \ldots, n\}$ back to itself. A neat way of explicitly defining it is by drawing a graph with vertices $\{1, 2, ..., n\}$ and for each $i \in \{1, 2, \ldots, n\}$ a directed edge $i \to f(i)$.

The set of all such bijections on $\{1, 2, \ldots, n\}$ under composition forms a *symmetric group* S_n.

Bijectivity implies that there's exactly one edge starting at each point and one ending at it, so the graph of f consists of disjoint cycles.

Given $f \in S_n$, let $(a_1\ a_2\ \ldots\ a_k)$ denote a cycle of f with $f(a_i) = a_{i+1}$, $f(a_k) = a_1$. Then each element $f \in S_n$ can be written in *disjoint cycle notation* as

$$f = (a_{1,1}\ a_{2,1} \ldots a_{k,1})\ (a_{1,2} \ldots a_{k,2}) \ldots (a_{1,j} \ldots a_{k,j})$$

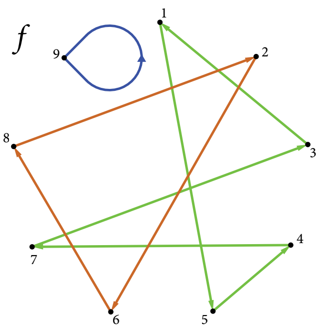
So $u = (15473)(268)(9)$. Let $v = (18594)(367)(2)$.

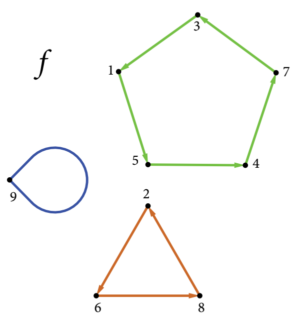
The graphs of u and v are *isomorphic*, i.e. there exists a bijection b of the vertices (which, of course, is itself an element of S_n).

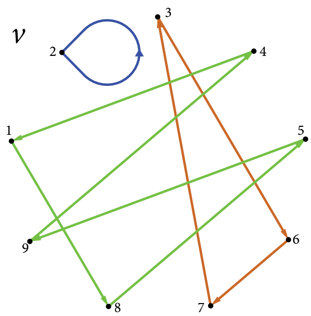
Cycle notation of functions gives us an isomorphism $b = (1)(2345879)(6)$. But we can check that in fact $u = b^{-1}vb$. So if we started with points in a circle labelled with consecutive numbers, relabelled them according to b, drew the edges according to v, and re-labelled them again, we'd get a graph that looks the same as graph of u. In fact:

Theorem Bijections $f, g \in S_n$ are conjugate iff their graphs are isomorphic. Moreover, $\theta : f \to g$ is a graph isomorphism iff $f = \theta^{-1} g\theta$.

Thus the conjugacy classes in S_n correspond to sets of bijections of the same cycle type.







Genus of a Graph

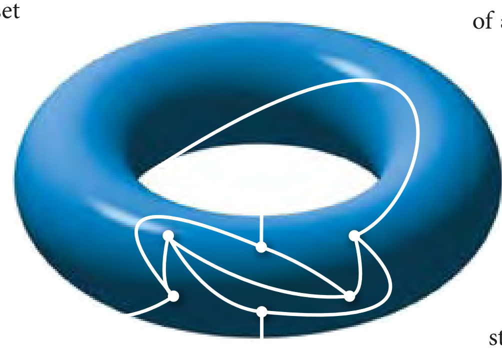
Loosely speaking, as any cycle can be drawn in a plane without its edges intersecting, so can any graph of a bijection. Graphs with this property are called planar. But which graphs are planar and what can happen if they're not?

Let G be a graph with vertices V and edges E and consider the set

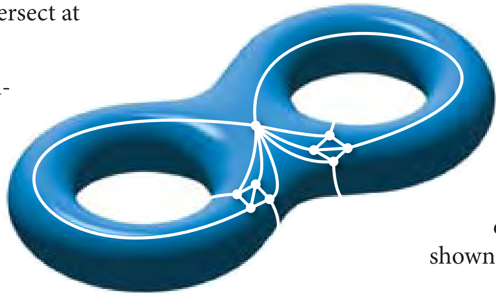
$$\mathcal{G} = \bigcup_{vw \in E} [0,1]_{vw},$$

where

1. for each $vw \in E$, arc $[0,1]_{vw}$ is homeomorphic to $[0,1]$, 0 being identified with v and 1 with w,
2. for each pair vw, $v'w' \in E$ of distinct edges, arc $[0,1]_{vw}$ and $[0,1]_{v'w'}$ intersect at most at vertices.



$\mathcal{G}$ with the above topology is a *topological representation* of a graph $G = (V,E)$. As in topology, we're mostly interested in graphs up to homeomorphism.



Drawing of a graph G is then a continuous mapping from $\mathcal{G}$ to the plane. If the mapping is also injective, we get a homeomorphism from $\mathcal{G}$ to a subset of the plane and we say that the graph is planar. It's easy to determine whether a graph is planar by using the following famous theorem:

Theorem (Kuratowski) A graph is planar iff it contains no sub-graph, homeomorphic to the complete graph K_5 or the complete bipartite graph $K_{3,3}$.

But as soon as two images of edges cross somewhere else than at vertices, the mapping is not injective and can not be a homeomorphism. Suppose we start with a drawing of a graph G. First notice that by modifying edges a bit we can make sure that at most two edges intersect any point, that is not a vertex. Now map a plane to a sphere via stereographic projection, and get a drawing of G on a sphere. Add an extra handle to a sphere for each intersection, and let one of the edges go over the other around the handle, so that they don't intersect. The result is an *embedding* of a graph G on surface $\mathcal{S}$, i.e. a homeomorphism e from $\mathcal{G}$ to a subspace of $\mathcal{S}$.

The *genus* of an orientable surface $\mathcal{S}$ is the number of handles added to a sphere to obtain it, but see the next section for more details. If the graph we started with was finite, we've added at most finitely many handles, and can then define the *genus of a graph* to be the smallest number of handles we need to add to a sphere to get an embedding of a graph on a surface. We've just shown the following:

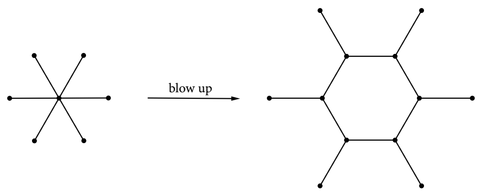
Theorem The *genus* of a finite graph exists.

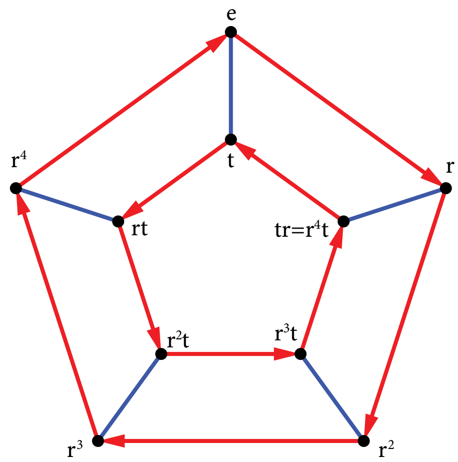
Surfaces

An embedding of a graph $\mathcal{G}$ on a surface $\mathcal{S}$ is a *triangulation* if any point $P \in \mathcal{S} \backslash \mathcal{G}$ lies in a region, bounded by exactly three edges of $\mathcal{G}$.

By a *surface* we mean a 2-manifold, that is a Hausdorff topological space X, such that every point of X has a neighbourhood, homeomorphic to an open ball in $\mathbb{R}^2$.

It can be shown that a triangulation of any compact surface exists, and playing with it in a clever way we can see that, more excitingly, in fact any compact connected surface is homeomorphic to a space obtained from a $2n$-gon by identifying its edges in pairs.





By considering the homology group of suitably reduced polygons and requiring that the surface is orientable, we can show that each surface is isomorphic to a g-holed torus.

Regular Graphs of Degree 3

A *regular graph of degree k* is a graph in which all each vertex has k neighbours.

Theorem There are infinitely many regular graphs of degree 3 of genus g for any g.

Outline of proof (see [5] for details): Given a triangulation of a surface of genus g, blow up each of the vertices as shown in the picture, to get a regular graph of degree 3, denote it G_1. To get G_2 blow up your favourite vertex of G_1. Proceed inductively to get G_n for any n.

Genus of a Group

A finite group G, generated by $g_1, g_2, \ldots, g_k$, can be represented by a *Cayley graph*, with vertices the elements of G and edges of colours $1, 2, \ldots, k$, where for each $g \in G$ and $i \in \{1, 2, \ldots, k\}$, there is an edge of colour i from g to $g_i g$.

Above you can see the Cayley graph of $D_5 = \langle\, r, t \mid r^2 = t^5 = e, rtr = t^{-1} \rangle$. Note that the Cayley graph of a cyclic group C_n is an n-cycle.

Thus there are exactly k edges starting at each of the vertices G and by considering $g_1^{-1}, g_2^{-1}, \ldots, g_k^{-1}$ we can conclude that any Cayley graph is regular.

Clearly the Cayley graphs of cyclic and dihedral groups are planar. Also the genus of any group generated by two elements is at most 1. Combined with results from section on regular graphs, one might expect that there are infinitely many groups of any given genus. However, the following is proved in [5]:

Theorem The number of groups of any genus g greater than 1 is finite.

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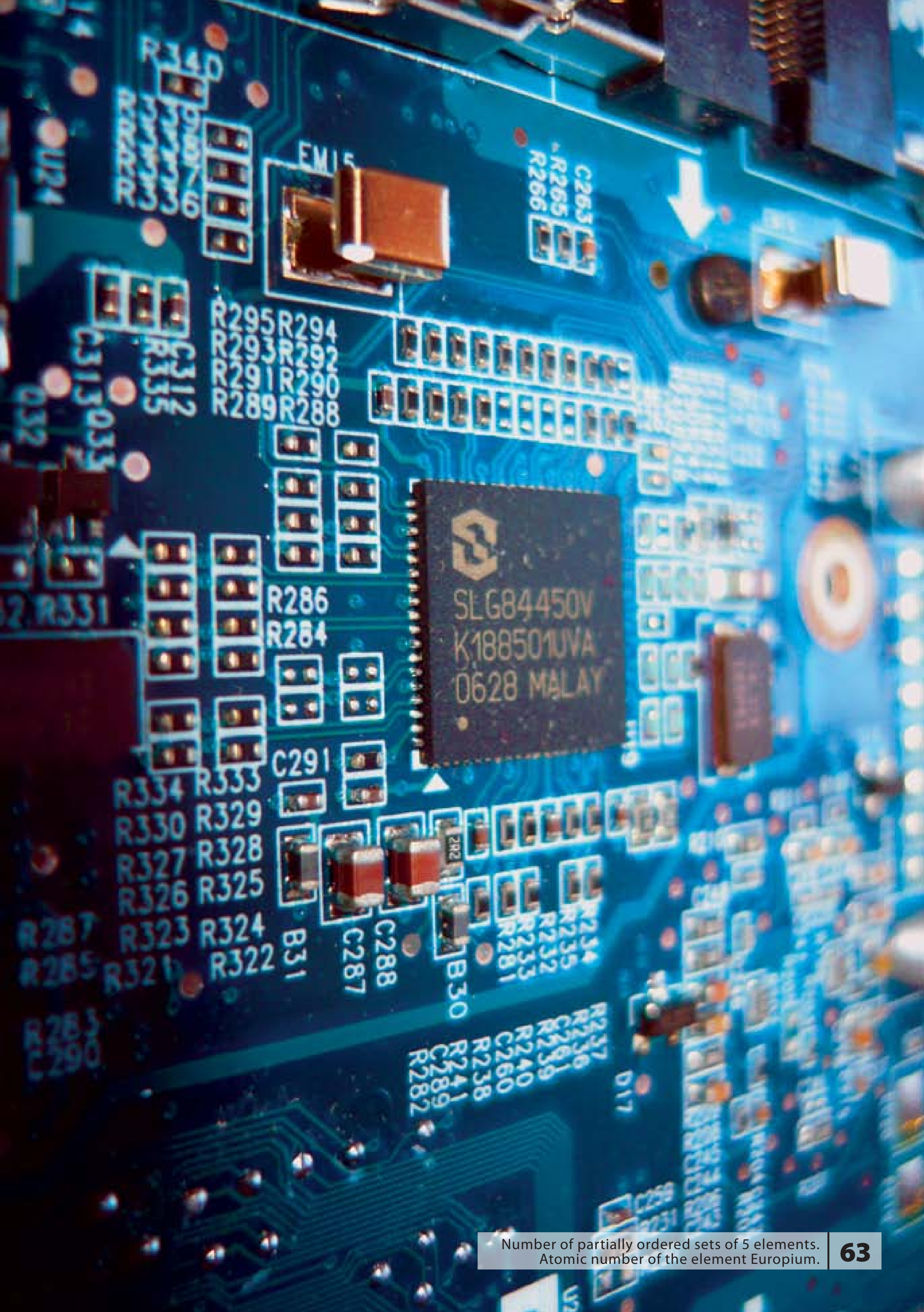
Graph theory is of fundamental importance ▶
when designing and optimising computer chips.

Only number whose cube, 238328, consists of three digits each occurring two times.



Number of partially ordered sets of 5 elements.
Atomic number of the element Europium.

Using the Finite Simple Groups

Cheryl Praeger, University of Western Australia

The finite simple group classification, announced by Daniel Gorenstein in February 1981, was one of the greatest triumphs of late twentieth century mathematics, and to this day its ramifications continue to drive cutting-edge developments across many areas of mathematics. The list of finite simple groups is surprisingly short: for each prime p, the cyclic group C_p of order p is simple; for each integer n at least 5, the group of all even permutations of a set of size n forms the simple alternating group A_n; there are finitely many additional infinite families of simple groups called *finite simple groups of Lie type*; and there are precisely 26 further examples, called the *sporadic simple groups*, of which the largest is the Monster (containing 808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000 elements!).

Already in 1981, some consequences of the classification were 'waiting expectantly in the wings'. For example, we immediately could list all the finite groups of permutations under which all point-pairs were equivalent (the 2-transitive permutation groups) [3].

Simple Groups and Algebraic Theory

For other problems it was unclear for a number of years whether the simple group classification could be applied successfully in their solution. One of the most famous of these was a 1965 conjecture of Charles Sims at the interface between permutation group theory and graph theory. It was a question about finite primitive permutation groups. The primitive groups form the building blocks for permutation groups in a somewhat similar way to the role of the finite simple groups as building blocks (composition factors) for finite groups. Sims conjectured that there is a function f on the positive integers such that, for a finite primitive permutation group in which a point stabiliser H has an orbit of size d, the cardinality of H is at most $f(d)$. In graph theoretic language: for a vertex-primitive graph or directed graph of valency d (each vertex is joined to d other vertices), there are at most $f(d)$ automorphisms (edge-preserving permutations) fixing any given vertex. Proof of the Sims conjecture [5] in 1983 required detailed information about the subgroup structure of the Lie-type simple groups, and was one of the first non-trivial applications of the finite simple group classification in Algebraic Graph Theory, see [6, Section 4.8C]. The new approach in [5] was later developed into a standard framework for applying the simple group classification to many problems about primitive permutation groups and vertex-primitive graphs.

Stunning new applications of the simple group classification in Algebraic Graph Theory continue to appear, and many new applications are accompanied by deep new results on the structure and properties of the simple groups. The most recent

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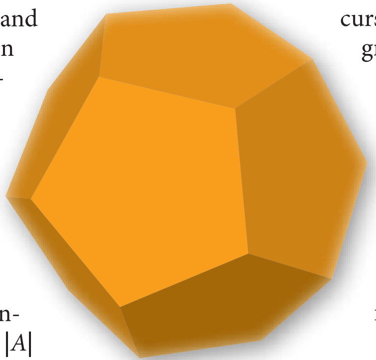
exciting developments relate to expander graphs. These are graphs or networks which are simultaneously sparse and highly connected. They have important applications for design and analysis of robust communication networks, for the theory of error-correcting codes, the theory of pseudo-randomness, and many other uses, beautifully surveyed in [11]. A family of finite graphs, all of the same valency but containing graphs of arbitrarily large size, is an *expander family* if there is a constant c such that the ratio $|\partial A| / |A|$ is at least c for every subset A of vertices of any of the graphs Γ in the family, where A contains at most half of the vertices of Γ and ∂A is the set of vertices of Γ at distance 1 from A. The new results confirm that many families of Cayley graphs for simple Lie-type groups of bounded rank are expander families. This flurry of activity began with a spectacular breakthrough by Helfgott [9] in 2008 for the two-dimensional projective groups $\mathrm{PSL}(2,p)$ over fields of prime order p. The strongest current results for bounded rank Lie type groups are consequences of new results for 'growth in groups' by Pyber and Szabo [19], and independently by Breuillard, Green and Tao [2] for the finite Chevalley groups.

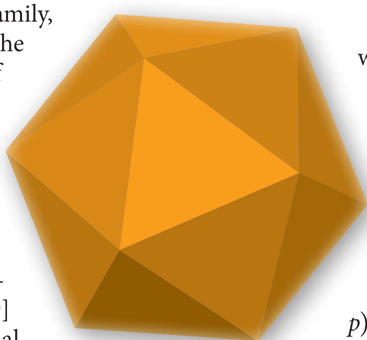
Simple Groups, Primes and Permutations

Several results about permutation groups have 'simple' statements making no mention of simple groups, but their only known proofs rely on the simple group classification, often on simple group theory developed long after the classification was announced. In fact many recent results in this area demand a deep and subtle understanding of the finite simple groups, especially their subgroup structure, element statistics, and their representations.

A surprising link between the number of primes and the finite simple groups was discovered soon

The symmetry groups of these platonic solids are finite simple. ▼





after the announcement of the simple group classification. It is a result due to Cameron, Neumann and Teague [4] in 1982. Each positive integer $n \geq 5$ occurs as the index of a maximal subgroup of a simple group, namely the simple alternating group A_n has a maximal subgroup A_{n-1} of index $|A_n| / |A_{n-1}| = n$. Let's call n a *maximal index* if $n = |G| / |H|$ for some non-abelian simple group G and maximal subgroup H with $(G, H) \neq (A_n, A_{n-1})$. It was proved in [4] that

$$\frac{\max(x)}{\pi(x)} \to 1 \quad \text{as} \quad x \to \infty,$$

where $\max(x)$ is the number of maximal indices at most x and $\pi(x)$ is the number of primes at most x. The limiting density of the set of maximal indices is 'explained' by the fact that, for each prime p, the projective group $\mathrm{PSL}(2,p)$ acts primitively on the projective line $\mathrm{PSL}(1, p)$ of size $p + 1$, and so has a maximal subgroup of index $p + 1$. The major motivation that led to this result was its consequence for primitive permutation groups, also proved in [4]: the number $D_{\mathrm{prim}}(x)$ of integers $n \leq x$ for which there exists a primitive permutation group on n points (that is, of *degree* n), other than S_n and A_n, satisfies $D_{\mathrm{prim}}(x) / \pi(x) \to 2$ as $x \to \infty$. Beside the primitive actions of $\mathrm{PSL}(2, p)$ of degree $p + 1$, the cyclic group C_p acts primitively of degree p, thus accounting for the limiting density ratio 2.

Two decades later I extended this result with Heath-Brown and Shalev in [8] as part of our investigation of quasiprimitive permutation groups, a strictly larger family of permutation groups than the primitive groups and important in combinatorial applications. (A permutation group is quasiprimitive if each of its nontrivial normal subgroups is transitive. Each primitive permutation group has this property, and so do many other permutation groups.) The crucial quantity we needed, in order to determine the behaviour of the degrees of quasiprimitive permutation groups, turned out to be the number $\mathrm{sim}(x)$ of *simple indices* at most x, where by a simple index we mean an index $|G|\,/\,|H|$ of an arbitrary subgroup H of a non-abelian simple group G such that $(G,H) \neq (A_n, A_{n-1})$. We proved that $\mathrm{sim}(x)\,/\,\pi(x)$ also approaches a limit as $x \to \infty$, and we proved that this limit is the number

$$h = \sum_{n=1}^{\infty} \frac{1}{n\,\phi(2n)} = 1.763085\ldots,$$

where $\varphi(m)$ is the Euler phi-function, the number of positive integers at most m and coprime to m. The analogous consequence (which had been our principal motivation for studying $\mathrm{sim}(x)$) was that the ratio $D_{\mathrm{qprim}}(x)\,/\,\pi(x)$ of the number $D_{\mathrm{qprim}}(x)$ of degrees $n \leq x$ of quasiprimitive permutation groups, apart from S_n and A_n, to $\pi(x)$ approaches $h+1$ as $x \to \infty$. In this case also, these ratios are accounted for by various subgroups of the simple groups $\mathrm{PSL}(2,p)$.

My ‘all-time favourite’ example of a deep result with a deceptively uncomplicated statement is due to Isaacs, Kantor and Spaltenstein [12] in 1995: let G be *any group of permutations* of a set of size n and let p be any prime dividing the order $|G|$ of G (that is, the cardinality of G). Then there is at least one chance in n that a uniformly distributed random element of G has a cycle of a length that is a multiple of p. The hypotheses of this result are completely general, giving no hint that the assertion has anything at all to do with simple groups. However the only known proof of this result relies on the finite simple group classification, and in particular uses subtle information about maximal tori and Weyl groups of simple Lie-type groups. These techniques were the same as those introduced in 1992 by Lehrer [13] to study the representations of finite Lie-type groups.

I recently worked with Alice Niemeyer and others to understand the precise conditions needed for this approach to be effective. We developed an estimation method in [16] and used it to underpin several Monte Carlo algorithms for computing with Lie-type simple groups (in [14], [15]). It produces sharper estimates for the proportions of various kinds of elements of Lie-type simple groups than alternative geometric approaches.

Simple Groups and Involutions

One of the first hints that understanding the finite simple groups might be a tractable problem was the seminal ‘Odd order paper’ of Feit and Thompson [7] in 1963 in which they proved that every finite group of odd order is soluble, or equivalently, that every non-abelian finite simple group contains a non-identity element x such that $x^2 = 1$. Such an element is called an *involution*, and the Feit–Thompson result, that each non-abelian finite simple group contains involutions, had been conjectured more than 50 years earlier by Burnside in 1911. The centraliser of an involution x consists of all the group elements g that centralise x in the sense that $xg = gx$. The involution centralisers in finite simple groups are subgroups that often involve smaller simple groups. Several crucial steps in the simple group classification involved systematic analyses of the possible involution centralisers in simple groups, resulting in a series of long, deep and difficult papers characterising the simple groups containing various kinds of involution centralisers.

Some important information about the simple groups can be found computationally, and key for this are efficient methods for constructing their involution centralisers. To construct an involution, one typically finds by random selection an element of even order that powers up to an involution, then uses Bray’s ingenious algorithm [1] to construct its centraliser. This worked extremely well in practice for computing with the sporadic simple groups. A more general development of Bray's method into proven Monte Carlo algorithms for Lie-type simple groups over fields of odd order required delicate estimates of various element proportions in simple groups – first given in a seminal paper of Parker and Wilson [17] (available as a preprint for several years before its publication), and then in full detail in [10]. The estimates and complexity analysis give a lower bound on the algorithm performance, but do not match the actual (excellent) practical performance. A major program is in train to find a realistic analysis and the first parts have been completed [14], [18].

The classification of the finite simple groups was a watershed for research in algebra, combinatorics, and many other areas of mathematics. It changed almost completely the problems studied and the methods used. To realise further the power of the classification for future applications, new detailed information is needed about the simple groups – and this will be gained both as new theory and through new computational advances.

Sum of the first 11 integers. Largest number below 2000 not containing the letter ‘e’ (an eban number).

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About the Author

Professor Cheryl Praeger is an Australian Research Council Federation Fellow and Director of the Centre for Mathematics of Symmetry and Computation. She received BSc and MSc degrees from the University of Queensland, a DPhil from the University of Oxford in 1973, and has received honorary doctorates from the Prince of Songkla University in Thailand in 1993 and the Universite Libre de Bruxelles in Belgium in 2005. Her research has focussed on the theory of group actions and their applications in Algebraic Graph Theory and for Combinatorial Designs; and algorithms for group computation including questions in statistical group theory and algorithmic complexity.

Professor Praeger is ARC Professorial Fellow, member of the Order of Australia, Fellow of the Australian Academy of Science, Executive Committee member of the International Mathematical Union, member of the Australian Research Council College of Experts, foundation board member of the Australian Mathematics Trust, Chair of the Australian Mathematics Olympiad Committee, former president of the Australian Mathematical Society, former Chair of the Australian Council of Heads of Mathematical Sciences, and was 2009 Western Australian Scientist of the Year.

Sequences

Trevor Pinto, Queens' College

1,3,7,9,13,15,21, 25, 31, 33, 37, 43, 49, 51, ...

Lucky Numbers

Do not be fooled by the title – this sequence is so much more than numerological nonsense. An alert reader may notice a similarity with the primes; this, however, is not nonsense: the process for generating lucky numbers is analogous to the 'Sieve of Erathosenes' method for generating primes.

Start with all positive integers: 1, 2, 3, 4, 5, 6, 7, 8, 9, … The first lucky number is 1. Then we remove every second number from the list: 1, 3, 5, 7, 9, 11, 13, … Now 3 is the smallest number on the list that is not already lucky, so we say 3 is lucky, and remove every third number from the list: 1, 3, 7, 9, 13, 15, … Now 7 is the smallest number on the list not already called lucky. Thus 7 is lucky itself, and we continue by removing every seventh number from the list. All the numbers that survive by the end are called Lucky numbers.

The similarities between lucky numbers and primes are so great, that lucky numbers satisfy the Prime number theorem (i.e. they have the same asymptotic density as the primes), and are conjectured to satisfy analogies to both the Twin Prime conjecture and the Goldbach conjecture.

by Ray Demski

1,2,2,1,1,2,1,2,2,1,2,2,1,1,2,1,1,2,2, ...

Kolakoski's sequence

Some sequences are defined as the collection of numbers with interesting properties, but Kolakoski's sequence defines itself by its properties: it is the unique sequence, starting with 1, that consists of 1's or 2's only, and is equal to its own runsequence. That is, the nth digit is also the length of the nth block (a block is a group of adjacent numbers that are equal). This property is fairly neat in itself, but the sequence has other interesting quirks.

For example, it may be generated by simple block substitution. Ignore the leading 1, and start with 22. The remainder of the sequence may be generated by the substitution rules: $22 \to 2211$, $21 \to 221$, $12 \to 211$, $11 \to 21$ Equally surprising, is the existence of a simple recursive formula for the sequence: If $K(n)$ is the nth term of the sequence, then

$$K\big(K(1)+K(2)+K(3)+\cdots+K(n)\big)=\frac{3+(-1)^k}{2}$$
$$K\big(K(1)+K(2)+K(3)+\cdots+K(n)+1\big)=\frac{3-(-1)^k}{2}$$

Although not obvious at first sight, this is enough to generate the entire sequence as $K(n)$, with no number bigger than 2 for all n.

Perhaps as impressive as both these facts are the things we don't know about the sequence. For instance, it is unknown whether the proportion of 1's in the sequence is 1/2; in fact it is not even known if the proportion exists.

Sequences are one of the simplest structures in mathematics, being just an ordered list of objects. There are many different kinds of sequence but this article deals only with integer sequences - fans of exact sequences, dance sequences or DNA sequences, I'm sorry to disappoint. Despite this restriction, the immense variety of such sequences ensures they are an important and interesting topic. Indeed, they even have their own journal, the creatively titled 'Journal of Integer Sequences'. This article is a collection of a few of the best.

by Jason Halayko

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, ...

Partition numbers

This arises from one of the simplest of combinatorial problems: for any natural number n, $p(n)$ is the number of ways of writing it as a sum of positive numbers (the order of these numbers is ignored). For example 3 may be expressed in three ways: 3, $1 + 2$ and $1 + 1 + 1$, so $p(3) = 3$.

The generating function for partition numbers factors elegantly:

$$\sum_{i=0}^{\infty} p(n)\, x^n = \prod_{k=1}^{\infty} \left(\frac{1}{1 - x^k} \right)$$

To see that this equality holds, expand each term in the product as a geometric series:

$$(1 + x + x^2 + x^3 + \cdots)\,(1 + x^2 + x^4 + x^6 + \cdots)\,(1 + x^3 + x^6 + x^9 + \cdots)\,\cdots$$

The coefficient of x^n in the product counts the number of ways of writing $n = a_1 + 2\,a_2 + 3\,a_3 + \cdots$ for non-negative integers a_i, that is the number ways of writing n as $n = (1 + 1 + \cdots + 1)\,(2 + 2 + \cdots + 2) + \cdots$ where each number i appears a_i times. Clearly, this is equivalent to the definition of the partition number.

Another surprising pattern in the partition numbers was discovered by Srinivasa Ramanujan, who proved the following congruences:

$$p(5n + 4) \equiv 0 \pmod 5, \qquad p(7n + 5) \equiv 0 \pmod 7, \qquad p(11n + 6) \equiv 0 \pmod{11}.$$

These are made more remarkable by the fact that no similar congruence exists for any other prime number.

Although no simple exact formula for partition numbers is currently known, Hardy and Ramanujan proved the following asymptotic formula (don't be surprised by the appearance of those ubiquitous numbers, π and e):

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$$

Smallest "weird" number. National speed limit, im mph, in the UK.

by Ray Demski

1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, 31131211131221, ...

Look and say sequence

Every term in this sequence (apart from the first one) is produced by reading the previous term. For instance, the fifth term is 111221, which can be read as 'three 1's, followed by two 2's, then one 1', making the next term 312211.

This sequence is often used as a 'guess the next term' puzzle, designed to trip up mathematicians due to its apparently non-mathematical recurrence relation, yet perhaps surprisingly, there are a wealth of mathematically interesting facts about the sequence.

For instance, every term ends in one, and no digit over 3 ever gets used (can you see why this is?). Also, the word lengths exhibit a pattern: the nth root of the length of the nth number tends to a limit, namely 1.303577…, which has been proved to be an algebraic number of degree 71. This is true regardless of what the first term is, except for one degenerate case, in which the starting term repeats ad infinitum. (Can you find this term? It has only 2 digits).

Most amazing of all is Conway's Cosmological Theorem: no matter what the starting value for the sequence is, it eventually splits into a sequence of 'elements' which don't interact with their neighbours in later terms of the sequence. (There are exactly 94 such elements, named Hydrogen, Helium, …, Plutonium by Conway).

by Marcio Abe

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, …

The greatest sequence of them all…

Even the most unremarkable of sequences may have hidden depths; 'Moessner's magic' is a prime example. First let us underline every second natural number. Then, in a new row, let us write down cumulative total of the *non-underlined* numbers:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1		**4**		**9**		**16**		**25**		**36**		**49**		**64**		**81**		**100**	

This gives us a neat method of generating the square numbers from the natural numbers, using only addition. But the real magic is yet to come…

In the first row, let us underline every third number. In the second row, let us underline every number that precedes a crossed out number in the previous row. Again we add a third row with the cumulative sums of the uncrossed numbers in the previous row.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	3		7	12		19	27		37	48		61	75		91	108		127	147
1			**8**			**27**			**64**			**125**			**216**			**343**	

This leads us to produce the cube numbers, again using no operation other than addition. Different initial configurations of underlined numbers leads to a different sequence (no prizes for guessing what underlining every nth number produces!). Perhaps the most surprising case of Moessner's magic is caused by crossing out the triangle numbers in the first row: we produce the factorial numbers:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	2		6	11		18	26	35		46	58	71	85		101	118	136	155	175
			6			24	50			96	154	225			326	444	580	735	
						24				120	274				600	1044	1624		
										120					720	1744			
															720				

Maximum number of spheres that can touch each other in a lattice packing in 6 dimensions.

by Jason Halayko

Further Reading

A good place to start is an article "Some very interesting sequences" by Conway and Hsu, available online, which contains the Look-and-Say sequence, and Moessner's magic in more detail, as well as a few other sequences.

A far more comprehensive source is the Online Encyclopedia of Integer Sequences, www.oeis.com, a searchable database with around 200 000 entries, which sticks rigidly to the definition of an integer sequence as any list of integers, not just special ones. It contains almost every integer sequence imaginable, from the interesting and useful, for example the number of groups of size n, to the dull and trivial, eg. the zero sequence and the list of stops on New York's No. 6 bus.

Those wishing for proofs and detailed studies of sequences should consult the Journal for Integer Sequences, also available online. Most interesting sequences are also discussed in both Wolfram MathWorld and Wikipedia.

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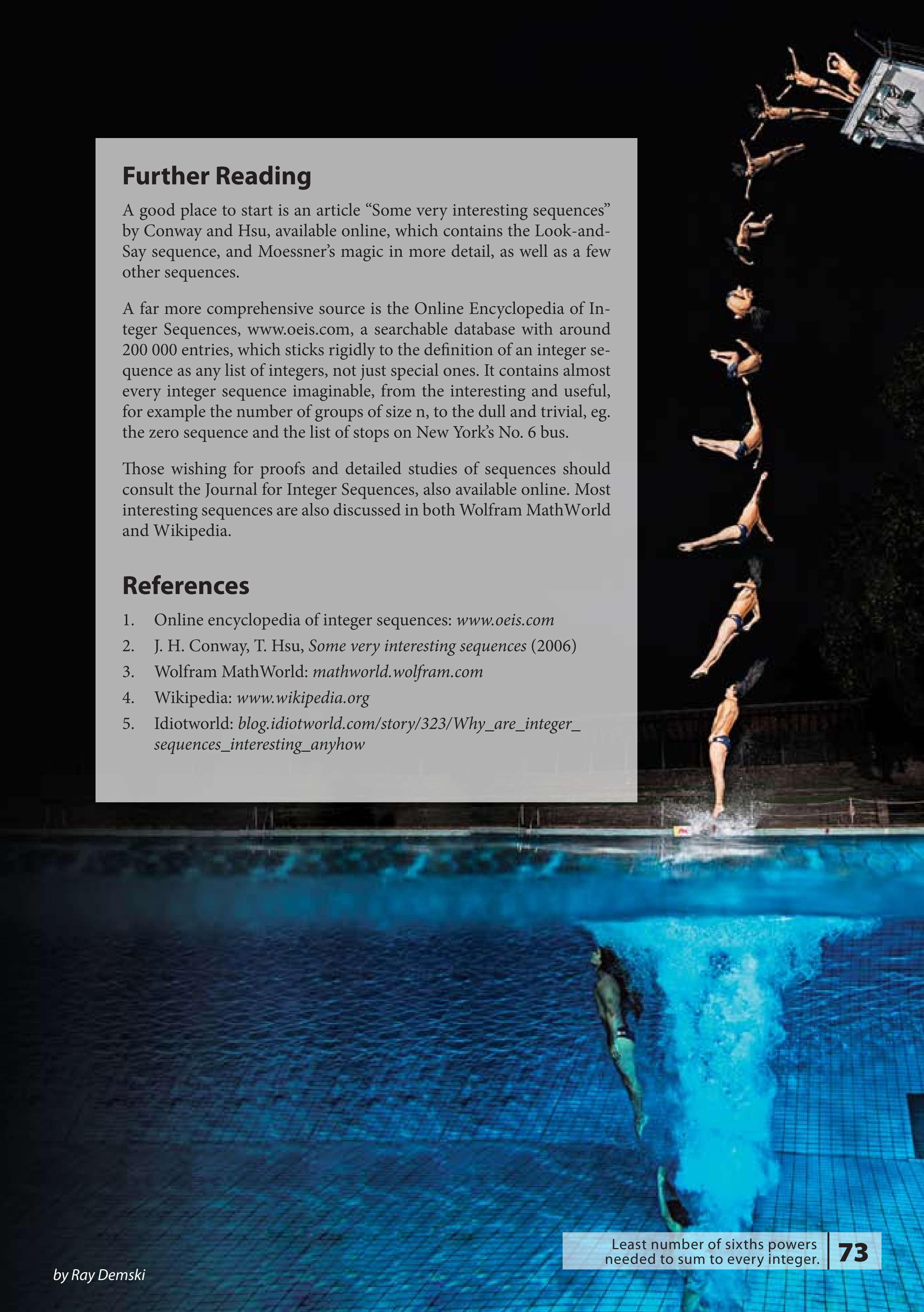

by Ray Demski

The Factors of Binomials

Samin Riasat, Queens' College

On a usual day, let's assume you are thinking about big numbers, say factorials, and digits, say zeros. A common question that might pop up in your mind could be, *how many zeros are there at the end of 1000 factorial ?*

You may already know the answer. I certainly do, it's 249, because I read it in a book a long time ago and for some weird reason it hasn't left my mind since then. You may also know how to find it. But just in case you don't, here's how to do it.

To find the number of zeros, we just need to find how many times the factors 2 and 5 appear in the prime factorization of 1000!. But there is already a large supply of factors of 2, so we need only find how many times the factor 5 appears. That's easy, there are $\lfloor 1000/5 \rfloor = 200$ multiples of 5, each contributing at least a 5 in the prime factorization of 1000!. But hang on, there are also factors of $5^2 = 25$ each of which contributes an extra 5. There are $\lfloor 1000/5^2 \rfloor = 40$ such factors. Similarly, each factor of $5^3 = 125$ contributes even one more 5's, and each factor of $5^4 = 625$ again contributes one more 5. We don't need to worry about 5^5 as there are no multiples of 5^5 between 1 and 1000. Thus our answer is

$$\left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{5^2} \right\rfloor + \left\lfloor \frac{1000}{5^3} \right\rfloor + \left\lfloor \frac{1000}{5^4} \right\rfloor$$
$$= 200 + 40 + 8 + 1 = 249.$$

After coming this far, you are probably thinking about generalising this. Indeed, it very easily generalises to the following theorem attributed to Legendre [1]. Henceforth, n and r are positive integers with $r \le n$ and p is a prime. We will also say the *exponent* of p in x to mean the exponent of p in the prime factorization of x.

Theorem 1 The exponent of p in $n!$ is given by

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor .$$

Why do we sum up to infinity? More importantly, why should p be prime? I shall leave these to you to worry about.

So now we know how to find the exponent of p in $n!$. We can do more and find the exponent of p in $n!/r!$, or even better, $n!/r!(n-r)!$, the binomial coefficient $\binom{n}{r}$. Binomial coefficients are important numbers, so it should be worth investigating their prime factors.

Proposition 2 The exponent of p in $\binom{n}{r}$ is

$$\sum_{j=1}^{\infty} \left(\left\lfloor \frac{n}{p^j} \right\rfloor - \left\lfloor \frac{r}{p^j} \right\rfloor - \left\lfloor \frac{n-r}{p^j} \right\rfloor \right).$$

There is an advantage in writing the sum in the above form.

Proposition 3 For real numbers a and b,

$$\lfloor a+b \rfloor - \lfloor a \rfloor - \lfloor b \rfloor \in \{0,1\}.$$

Proof: Writing $\{x\} = x - \lfloor x \rfloor$ we need to show that $\{a+b\} - \{a\} - \{b\} \in \{0,1\}$. Note that $\{a+b\} = \{\{a\} + \{b\}\}$. Thus if $\{a\} + \{b\} < 1$, then $\{a+b\} = \{a\} + \{b\}$. Otherwise $1 \le \{a\} + \{b\} < 2$, so $\{a+b\} + 1 = \{a\} + \{b\}$. $\square$

Hence it follows that the value of each bracket in the sum in **Proposition 2** is equal to either 0 or 1. Therefore the exponent of p is precisely the number of brackets equal to 1. From the above proof,

$$\left\lfloor \frac{n}{p^j} \right\rfloor - \left\lfloor \frac{r}{p^j} \right\rfloor - \left\lfloor \frac{n-r}{p^j} \right\rfloor = 1$$

$$\text{if and only if } \left\{ \frac{r}{p^j} \right\} + \left\{ \frac{n-r}{p^j} \right\} \geq 1.$$

Although this is a good criterion to check whether a prime divides a binomial coefficient, it is rarely used in practice. Nevertheless, we can deduce some quick facts from this: for example, if r and $n - r$ are both odd, then $\left\{\frac{r}{2}\right\} + \left\{\frac{n-r}{2}\right\} = \frac{1}{2} + \frac{1}{2} = 1$. This proves

Proposition 4 If n is even and r is odd, then $\binom{n}{r}$ is even.

In fact, we can do better.

Proposition 5 If the exponent of 2 in n is greater than the exponent of 2 in r, then $\binom{n}{r}$ is even.

Proof: Let $r = 2^q a$, $n = 2^q b$, with a odd and b even. Then $n - r = 2^q(b - a)$, and since $b - a$ is odd,

$$\left\{ \frac{r}{2^{q+1}} \right\} + \left\{ \frac{n-r}{2^{q+1}} \right\} = \left\{ \frac{a}{2} \right\} + \left\{ \frac{b-a}{2} \right\} = \frac{1}{2} + \frac{1}{2} = 1.$$

□

We can use our ideas to investigate the odd and even entries in Pascal's triangle. If we denote the odd and even entries by black dots and blanks, respectively, we will get a beautiful fractal called the Sierpinski triangle [2].

But first let's go back to **Proposition 2** and see if we can find a better criterion. There are powers of p everywhere**,** which suggests us to look at things in base p. Let the base p representations of n, r and $n - r$ be

$$n = \overline{a_k a_{k-1} \ldots a_1 a_0} = a_k p^k + \cdots + a_1 p + a_0$$
$$r = \overline{b_k b_{k-1} \ldots b_1 b_0} = b_k p^k + \cdots + b_1 p + b_0$$
$$n - r = \overline{c_k c_{k-1} \ldots c_1 c_0} = c_k p^k + \cdots + c_1 p + c_0$$

where $a_k \neq 0$. Afer some simple manipulations we find that

$$\left\lfloor \frac{n}{p^j} \right\rfloor - \left\lfloor \frac{r}{p^j} \right\rfloor - \left\lfloor \frac{n-r}{p^j} \right\rfloor = 1$$

if and only if in base p,

$$\overline{1a_{j-1} \ldots a_1 a_0} = \overline{b_{j-1} \ldots b_1 b_0} + \overline{c_{j-1} \ldots c_1 c_0}.$$

We have just proven the following amazing result:

Theorem 6 The exponent of p in $\binom{n}{r}$ is equal to the number of carries when adding r and $n - r$ in base p.

This result was proven by Kummer in 1852 [3]. Here are some straightforward consequences.

1. $\binom{m+n}{m}$ is odd if and only if, for each i, the ith binary digit of m or n is 0.
2. p divides $\binom{p^k+n}{n}$ if and only if the $(k + 1)$th digit of n in base p is $p - 1$.
3. p divides $\binom{p^k-1+n}{n}$ if and only if p^k does not divide n.

Many amazing properties of the Sierpinski triangle can be deduced from our discussion so far. But I should better stop and not ruin the fun for you!

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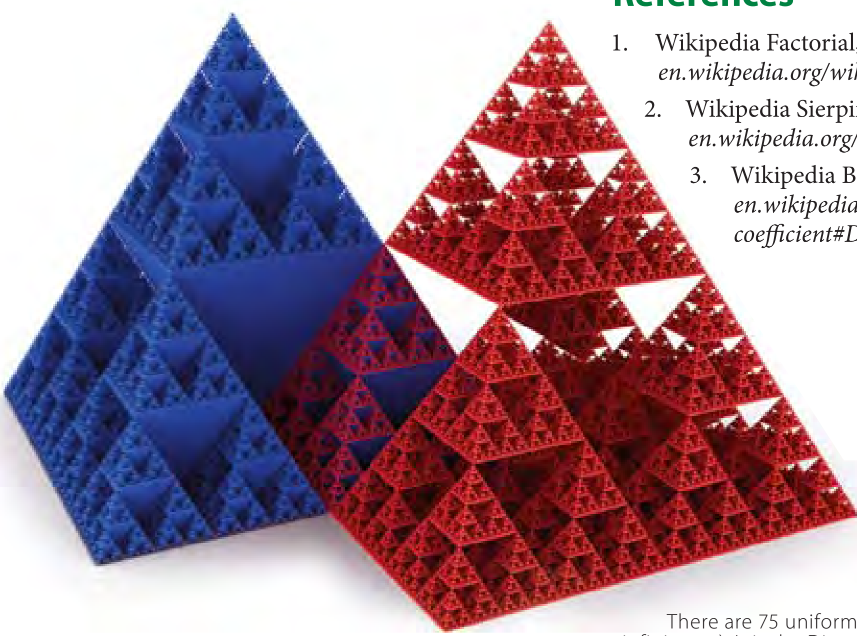


◀ A Sierpinski Pyramid (red) and it's inverse (blue).

Terrifying Trancendentals

Aled Walker, Trinity College

The year is 1873 and Charles Hermite is about to prove that the number e is transcendental. Before investigating the proof, let us set the scene.

A complex number z is algebraic if there exists a non-zero polynomial P with integer coefficients such that $P(z) = 0$. The numbers 3, 0.125 and $\sqrt{2}$, for example, are all algebraic.

A complex number z is transcendental if it is not algebraic, i.e. if there is no polynomial with integer coefficients having z as a root. Broadly speaking, *real* transcendental numbers are 'very irrational irrationals'. For almost all of what follows, we will limit our discussion to the reals.

The story so far

The irrationality of e was proved by Leonard Euler in 1744. Johann Heinrich Lambert, upon proving that π was irrational in 1761, conjectured that both e and π were transcendental. Unfortunately, at the time no one knew whether transcendental numbers existed *at all*!

The breakthrough was in 1844, when Joseph Liouville noted that algebraic numbers have poor rational approximations and thereby discovered a whole class of transcendental numbers. In 1851, he published the first explicit example, Liouville's Number, $L = 0.1100010000000000000000010\ldots = \sum_1^\infty 10^{-n!}$.

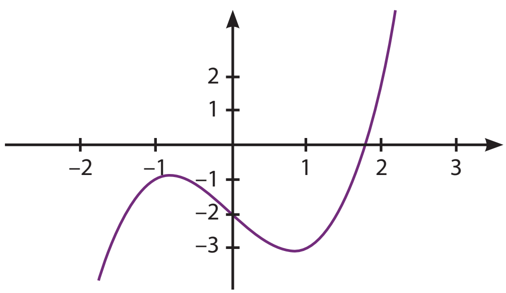
Liouville's Number

The proof is surprisingly simple, and boils down to the following imaginative argument.

1. Suppose L were algebraic. Consider the hypothetical polynomial P that we claim has L as a root.
2. Take a sequence s_m of rationals tending to L. Heuristically, $P(s_m)$ can't be *that* big. (The rigorous formulation of what 'not that big' really means is the bulk of the proof.)
3. However, if P has degree k and s_n has denominator q, then $|P(s_n)| \geq q^{-k}$.
4. Thus if s_n can get *really* close to L whilst maintaining a small denominator then at some point we will get a size contradiction.

Now L has been constructed via a very rapidly converging rational series, and the convergence is rapid enough to push through this argument.

Before you jump for joy, regrettably e does not yield directly to this rational approximation approach. We need new ideas.



Atomic number of the densest element, Osmium.
Number of books in the Catholic bible.

The Ideas

Again we start by assuming that e is algebraic.

Idea 1
Given that a size contradiction helped us with Liouville's number, it would be great if we could apply a similar notion here. Let our aim be to create some expression J with the property that $p! \le |J| \le c^p$, where c is a constant independent of p. Then, sending p to infinity, the factorial will 'beat' the power, and we will have a contradiction.

Idea 2
We know that e^x behaves well under integration and differentiation; it might therefore be pertinent to use integrals as bounding apparatus. Suppose that we could construct an integral $I(t) = e^t A - B(t)$, and have J be a clever sum of copies of $I(t)$ such that the e^t terms vanished due to our algebraic assumption. Suppose moreover that we could engineer the sum of the Bs to be necessarily 'quite big'. Then $|J|$ would also have to be 'quite big', since there are no e^t terms to mitigate the size of the Bs. But we could also try to bound $|J|$ from above using theory from integration. Then apply idea 1.

So let us suppose that $q_n e^n + q_{n-1} e^{n-1} + \cdots + q_1 e + q_0 = 0$, with integers q_i, $q_0 \neq 0$ and see where it takes us.

The Proof

The version given below is a simplification of Hermite's ideas, undertaken by Karl Weierstrass in 1885 and David Hilbert, Adolf Huritz and Paul Gordan in 1893.

Let $I(t) = \int_0^t e^{t-u} f(u)\, du$, where for the moment $f(u)$ is an arbitrary polynomial of degree m. This might seem hilariously random, but if we hit this with integration by parts repeatedly we get

$$I(t) = e^t \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t),$$

where $f^{(j)}(k)$ is the jth derivative of $f(x)$ evaluated at $x = k$. This is of the form we thought about before!

Three pioneers of Irrational numbers: Joseph ▶ Liouville (1809 – 1882), Karl Weierstrass (1815 – 1897) and Charles Hermite (1822 – 1901).

▲ Visualisations of the countable set of algebraic numbers in the complex plane. You can see the integers 0,1 and 2 at the bottom and +i near the top.

Left: Colours indicate the leading coefficient of the polynomial it is a root of (red = 1 i.e. the algebraic integers, green = 2, blue = 3, yellow = 4). Points becomes smaller as the other coefficients and number of terms in the polynomial become larger.

Right: Colours indicate degree of the polynomial the number is a root of (red = linear, i.e. the rationals, green = quadratic, blue = cubic, yellow = quartic). Points becomes smaller as the integer polynomial coefficients become larger. *Stephen Brooks*

Our clever sum is $J = q_n I(n) + q_{n-1} I(n-1) + \cdots + q_1 I(1) + q_0 I(0)$. This kills all the e^t terms, and we are left with

$$J = -\sum_{k=0}^{n}\sum_{j=0}^{m} q_k f^{(j)}(k).$$

Thus we have a completely free choice of $f(x)$, but we do need to be able to get a grip on the size of J. This can be done via a divisibility argument.

Let $f(x) = x^{p-1}(x-1)^p(x-2)^p \dots (x-n)^p$, with p prime. Why should this function be useful? Well, $f(x)$ looks a little like $(x^n)^p$ if you tilt your head enough, which is going to help with our upper-bounding of J and implementing Idea 1. Also, the function is designed so that many derivatives (evaluated at $0, 1, 2, \dots, n$) will be zero. We might therefore be able to understand the divisibility properties of the sum of the above derivatives better.

The critical result is that $p!$ divides all but one term in the double sum. This is because, in the midst of a gigantic product rule explosion, we can see that, with k an integer such that $0 \le k \le n$, we have

- if $j \le p-2$, then $f^{(j)}(k) = 0$ for all k;
- if $j = p-1$, then $f^{(j)}(k) = 0$ for all $k > 0$;
- if $j \ge p$, then each term in our product rule expansion of $f^{(j)}(k)$ will either be 0, or have the $(x-k)^p$ term differentiated p times. Therefore, we will have the $p!$ produced by this repeated differentiation 'out front'. Thus, $p!$ will divide $f^{(j)}(k)$.

Hence, $p!$ definitely divides all but one term in J, with the one uncertainty being $j = p-1$ and $k = 0$.

In the massive product rule expansion of this derivative, all terms will be zero except the one that includes x^{p-1} differentiated $p-1$ times. Thus just consider the term $f^{(p-1)}(0) = (p-1)!\,(-1)^{np}\,(n!)^p$.

Let's make $p > n$. Then $(p-1)!$ divides $f^{(p-1)}(0)$ but p does not. So $(p-1)!$ divides J, but $p!$ does not. Therefore $|J| \ge (p-1)!$ – hooray!

Now all we need is an upper bound on J, and this will take the form of c^p based on our observation that $f(x)$ looks like $(x^n)^p$ from the correct angle. More rigorously, observe that for $0 \le t \le n$,

$$\begin{aligned} |I(t)| &\le \int_0^t |e^{t-u} f(u)|\, du \\ &\le t\, e^t\, t^{p-1}(t+1)^p(t+2)^p \cdots (t+n)^p \\ &\le t\, e^t\, (2n)^{(n+1)p-1}. \end{aligned}$$

Therefore

$$
\begin{aligned}
|J| \le\ & |q_n|\,|I(n)| + |q_{n-1}|\,|I(n-1)| + \cdots + |q_1|\,|I(1)| \\
& + |q_0|\,|I(0)| \\
\le\ & |q_n|\, n\, e^n (2n)^{(n+1)p-1} + \cdots + |q_1|\, e\,(2n)^{(n+1)p-1} \\
\le\ & \max\big(|q_i|\big)\, n^2 e^n (2n)^{(n+1)p} \\
\le\ & A\, c^p,
\end{aligned}
$$

with A and c constants independent of p.

Therefore, $(p-1)! \le |J| \le A\,c^p$, which gives a contradiction for p suitably large. QED!

Reflection

We can run the entire proof with $f(x)$ divided by $(p-1)!$. In this case, the final contradiction has J being a non-zero integer trapped by a sequence tending to zero. This smells similar to the proof that π is irrational.

We achieved exactly what we set out to do in ideas 1 and 2; the queer truth is that we almost never referred to e at all! We only used the differentiation properties of e^x, when expanding $I(t)$ by parts. It might not come as a surprise then that a more general result can be proved by very similar means; if x is algebraic and non-zero then e^x is transcendental. This was achieved by Lindemann in 1882.

And you know what that means? Well, $e^{i\pi} = -1$, so π algebraic $\Rightarrow$ $i\pi$ algebraic $\Rightarrow$ -1 transcendental, which is a contradiction. So π is transcendental!

Think about plane geometry. In a coordinate system that gives the starting points integer coordinates, all points constructible by a compass and straight edge have algebraic coordinates. In order to square the circle, the great problem from antiquity, we have to construct the length $\sqrt{\pi}$, which is transcendental. Hermite and Lindemann had solved a 2200 year old enigma.

Addendum

An interesting historical note: Hermite proved his result about e in 1873, a truly great achievement. However, just one year later in 1874, Cantor showed that ‘almost all’ numbers are transcendental through his work on countability. Proving that a particular given real was transcendental was still ferociously difficult, but the problem of existence had been triumphantly annihilated.

Impossible Integrals

Damian Reding, Trinity College

No doubt, when walking along the path through the world of mathematics you have come across one of the widely known facts which give away no clue as to why they are true. Examples might be the non-existence of a general solution formula in radicals to polynomial equations of degree 5 or higher, or, given any group, the existence of a topological space with this group as the fundamental group, not to mention Fermat's Last Theorem. The purpose of this article is to provide relief from one of such burdens: The fundamental theorem of calculus guarantees existence of antiderivatives to all continuous functions, though it is well-known but unclear that some of them are not expressible in terms of so called elementary functions (or rather not finitely expressible to avoid series expansions). Further, it is natural to ask for a necessary and sufficient condition for a function to belong to this class. For a certain range of functions, including the classical example $x \mapsto e^{x^2}$, the solution to this innocent sounding problem is wrapped in terminology and buried in the depths of a weird field called differential algebra, which can be roughly understood as algebra imitating analysis. Let us begin with a crash course.

Differentiation on General Fields

To breathe life into this concept we no longer regard differentiation as a limiting process, but as evaluating a map: A *derivation* of a field F is a map $\partial : F \to F$ satisfying for all $a,b \in F$

$$\partial(a+b) = \partial a + \partial b,$$
$$\partial(ab) = (\partial a)b + a\,\partial b \quad \text{(Leibniz Rule)}.$$

(No, a derivation cannot be a field homomorphism – for a very simple reason, can you see it?)

The pair (F,∂) is a *differential field*; applying ∂ is referred to as *differentiation* and the value ∂f is the *derivative* of $f \in F$. By playing around with the Leibniz rule, we quickly obtain the *quotient rule*

$$\partial\left(\frac{a}{b}\right) = \frac{(\partial a)b - a\partial b}{b^2}$$

for all $a,b \in F$ and $b \neq 0$.

To induce linearity as satisfied by the usual differential operator, we must identify F as a vector space over some field, but the idea to take F itself is bad, because differentiation will in general not commute with multiplication. In order to overcome this obstacle to linearity, we need the first term in the Leibniz rule to vanish, so we consider the subset $C(F) := \{c \in F \mid c = 0\}$. Using the quotient rule we easily see that this is a subfield, the *field of constants*, of F, and it pleases the eye to see that now differentiation becomes a $C(F)$-linear operator $F \to F$.

As an example, the rational functions $F \cong \mathbb{C}(z)$ with $\partial = d/dz$ form a DF with $C(F) \cong \mathbb{C}$.

A *differential field homomorphism*

$$(F, \partial_F) \to (E, \partial_E)$$

is a field homomorphism $\delta : F \to E$, which respects the differential structures in the sense that the following diagram commutes:

Days needed by Phileas Fogg to travel around the world.
The A380 is the world's largest passenger aircraft.

$$\begin{array}{ccc} F & \xrightarrow{\delta} & E \\ {\scriptstyle\partial_F}\downarrow & & \downarrow{\scriptstyle\partial_E} \\ F & \xrightarrow{\delta} & E \end{array}$$

A *differential extension*, written $E :_\partial F$, is formally an injective DFH $(F, \partial_F) \to (E, \partial_E)$, but in practice we assume F to be wlog a subfield of E with ∂_F the restriction of ∂_E to F. Of importance for the theory is the following (perhaps a bit surprising) result:

Existence of Differential Extensions Let (F, ∂_F) be a differential field and let $E : F$ be a field extension. Then there exists an extension of ∂_F to a derivation ∂_E of E.

Proof: (1) If $E = F(t)$ with t transcendental over F this is easy: We can naturally extend the derivation of F to a derivation $\bar{\partial}$ of $F[t]$ by setting:

$$\bar{\partial}\left(\sum_{i=0}^{n} a_i t^i\right) := \sum_{i=0}^{n} (\partial a_i) t^i$$

and then extend $\bar{\partial}$ to a derivation of the field of fractions $F(t) = E$ via the quotient rule.

If $E = F(a)$ with a algebraic over F we need a clever trick: We consider the derivation of $F[X]$ given by the formal derivative

$$\frac{d}{dX} : \sum_{i=0}^{n} a_i X^i \mapsto \sum_{i=1}^{n} i a_i X^{i-1}$$

and then note that for any $g(X) \in F[X]$ the linear combination $\partial_g := \bar{\partial} + g(X)\, d/dX$ is a derivation of $F[X]$ extending ∂_F ($\bar{\partial}$ being the derivation from (1), t now replaced by X). Let $p_a(X) \in F[X]$ be the minimal polynomial of a, so that $dp_a/dX(a) \neq 0$. Note that $F(a) = F[a]$ since a is algebraic over F, so we can choose $g \in F[X]$ such that

$$(\partial_g p_a)(a) = (\bar{\partial} p_a)(a) + g(a) dp_a/dX(a) = 0,$$

so we have $(\partial_g p_a)(a) = p_a f$ for some $f \in F[X]$. This shows that σ_g maps the ideal (p_a) into itself and so defines differential structure on the corresponding quotient

$$\begin{aligned} \partial : F[X]/(p_a) &\to F[X]/(p_a), \\ f + (p_a) &\mapsto \partial_g f + (p_a) \end{aligned}$$

But since evaluation $X \mapsto a$ gives rise to an isomorphism

$$F[X]/(p_a) \xrightarrow{\cong} F[a] = F(a) = E$$

there exists a derivation ∂_E on E extending ∂_F.

(3) Together, (1) and (2) show that every extension field of F obtained by adjoining finitely many elements admits a derivation extending ∂_F, and by appropriate use of Zorn's lemma the result extends to any extension $E :_\partial F$. Further, if we assume $E : F$ algebraic ($\Leftarrow$ finite), then uniqueness of ∂_E follows easily from the existence of minimial polynomials.

For the following section we note that the theorem enables us to speak automatically of a differential extension $E :_\partial F$ whenever we are given a field extension E of a differential field (F, ∂). Thus, we denote an extended and fixed derivation of E by ∂ as well.

The Nature of Derivatives

An element $e \in E$ is *elementary* over F if it is either

1. algebraic over F,
2. an exponential over F, so $\partial e/e = \partial f$ for some $f \in F$,
3. a logarithm over F, so $\partial e = \partial f/f$ for some $f \in F$.

A DFE obtained by adjoining finitely many elements elementary over F, is called an *elementary* DFE of F. Now, the key definition is the following: An $f \in F$ has an *elementary integral* if there exists an EDFE $E :_\partial F$ s.t. $\partial e = f$ for some $e \in E$. Of course every good theory has its big theorem:

Liouville-Ostrowski Theorem Let (F, ∂) be a DF of characteristics zero and let $f \in F$ have an elementary integral over some EDFE $E :_\partial F$ with the same field of constants. Then f is of the form

$$f = \partial v + c_1 \frac{\partial u_1}{u_1} + \cdots + c_n \frac{\partial u_n}{u_n}$$

for some $v, u_1, \ldots, u_n \in F$ and constants $c_1, \ldots, c_n \in C(F)$.

The essence of this is that Elements in a DF which are derivatives in EDFEs are derivatives in the base DF themselves,up to finitely many additive constant multiples of logarithmic derivatives.

The restriction to zero characteristic is necessary in order to be able to speak about linear independence of the above constants over rational numbers. The rather lengthy and intricate proof of this powerful theorem is by induction on the tower length of the EDFE and makes use of elementary *differential Galois theory*. It essentially consists of algebraically elaborate disposals of numerous cases by contradiction and all we can do here is to refer to M. Rosenlicht's article entitled *Liouville's Theorem On Functions With Elementary Integrals.*

Elementary Antiderivative if and only iffffff…

Equipped with Liouville's theorem, we now restrict consideration to the differential field of (real or) complex-valued rational functions $\mathbb{C}(z)$ equipped with the derivation given by the usual analytic derivative $\partial = d/dz$. In this special case the elementary integrals are referred to as *elementary antiderivatives*. Examples of EDFEs here are $\mathbb{C}(z, e^z)$ or $\mathbb{C}(z, \log z)$ and all the usual *elementary functions* known from Analysis I are obtained that way. Further, we note that the constants in every EDFE of $\mathbb{C}(z)$ will be precisely the constant functions, so the assumptions of Liouville's theorem are satisfied.

Existence of Elementary Antiderivatives Let f, g be rational functions of the (real or complex) variable z, such that f is non-zero and g is non-constant. Then, the function $z \mapsto f(z)\, e^{g(z)}$ has an elementary antiderivative if and only if there exists a rational function r satisfying the 1st order ODE

$$\frac{dr}{dz} + r\frac{dg}{dz} - f = 0$$

If namely $r \in \mathbb{C}(z)$ satisfies the ODE, then the antiderivative $z \mapsto r(z)\, e^{g(z)}$ does the job. Conversely, write $t = e^g$, $(.)' := d/dz$, and suppose fe^g has an elementary integral in some EDFE of $\mathbb{C}(z,t)$, so by Liouville's theorem there exist $c_1,\ldots,c_n \in \mathbb{C}$ and $u_1,\ldots,u_n,v \in \mathbb{C}(z,t)$ such that

$$f\,t = v' + \sum_{i=1}^{n} c_i \frac{u_i'}{u_i}$$

A field-theoretic argument establishes the transcendency of t over $\mathbb{C}(z)$ and consequently the proof of Liouville's theorem shows that $v \in \mathbb{C}(z)[t]$ and $u_i = g^i t^{k^i}$ for some $g^i \in \mathbb{C}(z)$ and $g^i \in \mathbb{Z}$. This gives

$$\begin{aligned} f\,t &= v' + \sum_{i=1}^{n} c_i \frac{(g_i t^{k_i})'}{g_i t^{k_i}} \\ &= v' + \sum_{i=1}^{n} c_i \left(\frac{g_i'}{g_i} + k_i \frac{t'}{t}\right) \\ &= v' + \underbrace{\sum_{i=1}^{n} c_i \left(\frac{g_i'}{g_i} + k_i g'\right)}_{=:\, c\, \in \mathbb{C}(z)}. \end{aligned}$$

Hence, there is $m > 0$ such that we can write $v = \sum_{i=1}^{m} v_i t^i$ with $v_i \in \mathbb{C}(z)$, $v_m \neq 0$. Differentiating gives

$$v' = \sum_{i=0}^{m} (v_i' t^i + i v_i t^{i-1} t') = \sum_{i=0}^{m} (v_i' + i v_i g') t^i$$

and for all i such that $v_i \neq 0$ we must have $v_i' + i\, v_i\, g' \neq 0$, otherwise $i\,g' = -v_i'/v_i$ would have only simple poles, which is a contradiction. Therefore $m = 1$, so we take $r = v_1$ and note

$$ft - c = v' = v_0' + (v_1' + v_1 g')t = v_0' + (r' + rg')t.$$

Transcendency of t implies $r' + rg' = f$, as required.

Showdown

Corollary Regrettably, the function $z \mapsto e^{x^2}$ has no elementary antiderivative.

This will not take a lot of believing anymore as we are sceptical of the existence of polynomials with poles: If an elementary antiderivative of $z \mapsto e^{x^2}$ existed, then by the above criterion we could pick non-zero polynomials p, q of degrees m, $n \geq 0$, respectively, such that $r = p/q$ satisfies $r' + 2zr = 1$. Substituting this gives

$$\frac{p'q - pq'}{q^2} + 2z\frac{p}{q} = 1$$

$$p'q - pq' + 2zpq = q^2$$

Comparing polynomial degrees gives $mn + 1 = n^2$, so $n = 1$ and $m = 0$. Hence $r'/r = 1/r - 2z = q/p - 2z$ is a polynomial in z. But r is non-constant, so r'/r must have a pole!

Very closely related to the above is Differential Galois Theory, a theory which roughly speaking does with linear differential operators over differential fields what usual Galois theory does with polynomials over algebraic fields. It evolves much in analogy to algebraic Galois theory and ends up answering questions like why the 2nd order ODE $y'' + xy = 0$ is insolvable in terms of elementary functions and integration, or why there does not exist a linear differential equation satisfied by $x \mapsto \sec(x)$.

Further Reading

For a nice informal account of DGT see Andy R. Magid's article *Differential Galois Theory*, for a more rigorous treatment refer to e.g. M. van der Put's and M. Singer's *Galois Theory of Linear Differential Equations*.

Engage. Explore. Excite. *Scientific Collaboration*

Dr Marj Batchelor, DPMMS

I owe my mathematical career to a turkey. I was at the time a PhD student at MIT, a very wayward one, following my own thoughts and heedless of my supervisor's suggestions. I had a shelf full of exciting ideas that had led nowhere. Attempt 32A for example, had demonstrated that the category of supermanifolds offered no advantages over vector bundles as a tool for distinguishing one smooth manifold from another. My friend and colleague, Barbara Peskin, had cooked Thanksgiving dinner for her brother, the now eminent physicist, and his friends. Inevitably, dinner table chatter rapidly turned to physics and stayed there, leaving my algebraic geometer friend somewhat out of the flow. She did however hear the word "supermanifold" mentioned repeatedly, and this offered a point of entry. "Oh, Marj says they are all just vector bundles." It rather stopped the conversation than furthered it; the result was significant news. Thus attempt 32A was rescued from oblivion, dusted off, typed and submitted as a thesis. Had it not been for that turkey, my career as a mathematician might have been short indeed.

That turkey also catapulted me, a dyed-in-the-wool (dyed on the sheep?) algebraist, into the land of theoretical physics. Looking back, I am amazed at my arrogance in supposing that I could aspire to learn enough field theory to make any useful contribution. Perhaps the natural conceit of youth was supplemented by the advantages of an American undergraduate education. Here, we spend hours in faculty meetings debating course syllabuses in the search for some unique (possibly space filling) path that will touch all fields of mathematics. In the US, universities despair of the existence of such a route, and offer modules to be selected at the students' whim, governed by a loose system of prerequisites. One consequence is that most US students have gaps in their mathematical background, which must be filled in the first few years of a PhD programme, but the system also confers a general confidence that anything can be learnt from books or colleagues and no fields are off limits. The US system even strongly encourages students to take courses outside their major. However it happened, the decision to study field theory and appoint myself as supermanifold mechanic to the quantum field theory and general relativity community seemed a natural step, certainly one I never questioned.

"I have come to appreciate how much the contact with students' interests well outside my personal mathematical comfort zone continues to influence the direction of my research."

Sharing Mathematical Interest

I never questioned the decision, I never even thought about it much. It is only in recent years, since I have had responsibility for PhD students in DPMMS, that I have begun to appreciate the true value of that diversion from established algebraic tracks. It is part of my job here to become familiar with what my PhD students are up to, and part of my job to sample the seminars given by Part III students during the Part III seminar

Number of 5-faces of a seven-dimensional hypercube.
Largest order of a permutation of 14 elements.

series. I have come to appreciate how much that contact with students' interests well outside my personal mathematical comfort zone continues to influence the direction of my research. I also have come to realise that not many others, students, post-docs or faculty, make use of the diversity of interest within the Centre for Mathematical Sciences (CMS) in Cambridge.

Most aspiring mathematicians correctly aim to train and work in a large department, with many others sharing their mathematical interests. The value of such an environment is beyond dispute. Nonetheless, the evident advantages of belonging to such an institution conceal a fatal flaw: there is often so much going on even within very narrow specification of subject that a conscientious student or post-doc can reach a mathematical saturation point while attempting to attend no more than those seminars and courses directly related to his or her own interests. It requires deliberate effort to make time and energy available to connect with colleagues in other fields.

I strongly recommend making this effort. I don't believe I possess any special genius that enables me to gain from external influences where others wouldn't. All in CMS could benefit. The opportunities are there, both within CMS and the university. The purpose of the collegial structure of the university is to provide smaller social groups in which pure mathematicians, engineers, and chemists might row together or share desks in a college orchestra. These colleagues are a major resource, an easy introduction to unfamiliar disciplines, and should be used as such. At CMS, colloquia, Part III seminars, even Happy Hour all provide opportunities to meet those working in different fields.

It is possible that some don't make the effort through despair of ever gaining enough expertise in a different field to make a useful contribution. Of course full familiarity with the new field is desirable, but the benefits of such external influences on research exist at all levels of understanding. I was not an apt student of physics. I never did achieve any deep understanding of field theory. I did listen lots, understanding perhaps only a small percentage of what was said. That small percentage was sufficient to generate research for over twenty years. More recently, under different external influences I have found myself toying with number theory and genetics. I will never be a number theorist or a geneticist, but aspects of the structure of the theory in those fields cause me to see the structures I have worked on in a fresh light, driving research in new directions.

"Whoever make these decisions has failed to see the importance of fundamental mathematical research. Mathematics provides the tools to think with, not just mathematical methods."

How Politics can determine the Direction of Research

This is very stimulating. Inevitably we do the research which excites us, and the bottom line of my argument in favour of diversification is that it is exciting. There are other important reasons why we might do well to diversify, and to be seen to diversify. A cashstrapped government is looking hard at its spending on research. I would draw your attention to the EPSRC website, specifically the guide for fellowship applications:

> ***We will not accept applications in areas outside of those identified below.***
> *Applicants should refer to the thematic programme priorities for additional details on the research areas we wish to appoint fellows in.*

The listing of areas supported within mathematics is short: statistics and probability. That's it, although theoretical physicists may still apply under physics [1]. This together with the emphasis on social and economic impact in judging the value of research is worrying [2]. Beyond the obvious advantage of rapid return on investments, I am not clear on the rationale for the strategy. It is not clear to me who had the final say in these decisions, politicians or mathematicians. It is not clear to me which answer to that question is more worrying. I should hope that my colleagues are not so short-sighted. I am even less comfortable with the thought that politicians are taking control of the direction of mathematical research. The negative impact of the policy seems very clear: these are decisions which will certainly force many of my best students to emigrate. It is difficult for me to understand how losing such able mathematicians will in any way enhance the strength of mathematics in this country.

Whoever make these decisions, if they have failed to see the importance of fundamental mathematical research, perhaps we must take some of the blame. We know that mathematics provides the tools to think with, not just mathematical methods. If this is not obvious to those who make decisions, the remedy is in our hands. Placing greater importance on explaining our interests to the man on the street will help. However, the most incontrovertible demonstration of the usefulness of pure mathematics is the recognition by scientists in other fields of the importance of our research in their subject. Engaging with those in other fields matters.

Allowing other fields to influence the direction of our research and being available to those working in other fields as a resource are both part of the job. Working in as diverse a place as CMS is a privilege, and as with all privileges, comes with the responsibility to use it well. I am writing this in case no turkey arrives to disrupt your lives and deflect your course of study and research. Diversify. It is part of the job. It may also save our jobs, but the bottom line is this: it is exciting and it is fun. Engage.

References

1. www.epsrc.ac.uk/funding/fellows/Pages/areas.aspx
2. www.hefce.ac.uk/research/ref/pubs/2011/01_11/01_11.pdf

Book Reviews

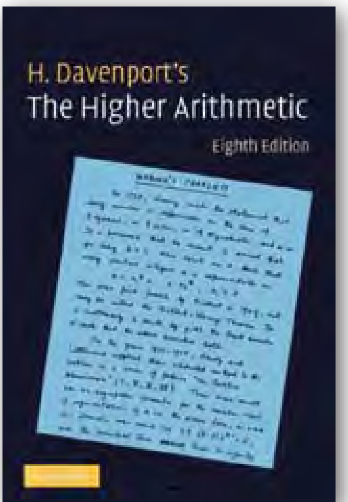
Algebra, Arithmetic, Anecdotes, Apologies…

A Measure of All Things

Ian Whitelaw *David & Charles, 2007*
ISBN 0-7253-2696-1 *£9.99*

Not just a collection of conversion tables but a chronicle of measurement and its units throughout history. This a delightful book to dip into at random: fascinating anecdotes feature on every page, illustrated with a pleasing minimalist aesthetic. Accounts are given of both the natural and cultural phenomena that led to the adoption of various measurement systems, as well as the philosophical and technical progress made in reaching those from which we benefit today.

Mathematics students will find the explanations of acceleration and related concepts redundant. Regardless, this book is excellent light reading and will make the perfect gift for those with casual interests in any science. *Sean Moss*



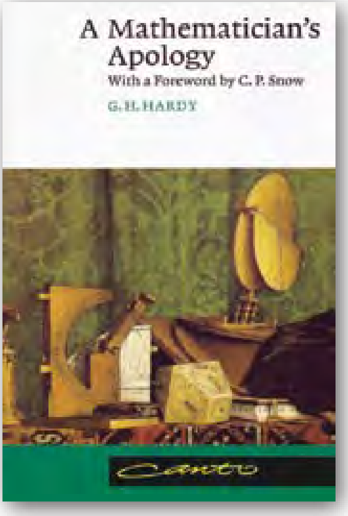
The Higher Arithmetic

Harold Davenport *Cambridge University Press, 8th edition 2008*
ISBN 0-521-72236-0 *£25.99*

It confesses to be an introduction to number theory, but this book still has plenty of substance. There is overlap with material in Part II Number Theory, but a book such as this would be perfect for a Part I mathematician interested in acquiring some of the elementary number theory missing from the early parts of the mathematical tripos.

The number of editions through which this book has been is a testament to its quality. Topics are covered in an efficient manner but not at the expense of clarity. The book is well-suited to studying at one's own pace: there are exercises, but in general they are not critical to the narrative, and there are many recommendations for further study on particular topics. *Sean Moss*

Only number known whose square has no isolated digits. Numbers of keys on a piano.



A Mathematician's Apology

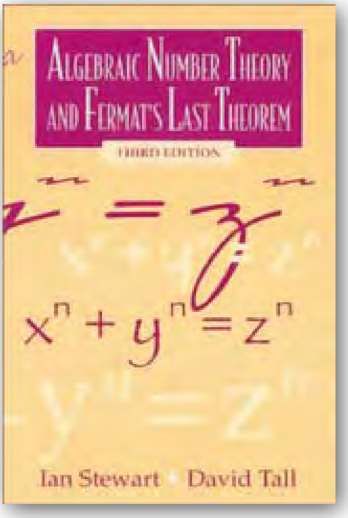
G.H. Hardy — *Cambridge University Press, reprinted 2008*

ISBN 0-521-42706-7 — *£14.99*

A rare glimpse inside the mind of creative genius. Underlying Hardy's "defence" of his life and work is really the outpouring of a lifetime's hopes and frustrations. The maths in it is basic but worthwhile, however, the book is a tremendous success for its sheer humanity. Mathematicians and non-mathematicians alike will find themselves drawn towards the author's personality and feeling sincerest empathy with him.

The foreword by C.P. Snow is not to be skipped – it takes up about half the volume – it does much to put Hardy's soliloquy into biographical and historical context. Very light on mathematical technicalities, this book is about the human interaction with mathematics. While we may recognize Hardy's sadness, the connection one feels to the mind of genius is ultimately most uplifting.

Sean Moss



Algebraic Number Theory and Fermat's Last Theorem

Ian Stewart, David Tall — *AK Peters, 2002*

ISBN 1-56881-119-5 — *£37.99*

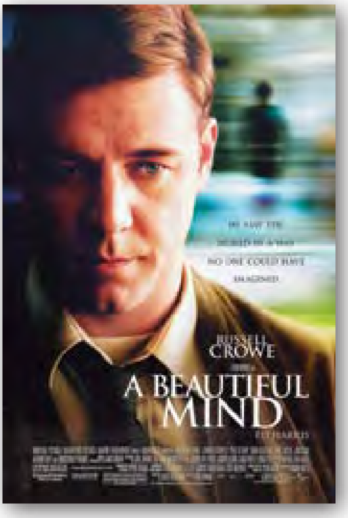
It is difficult to find a mathematics book that is both precise and informal. This book has both qualities, giving historical background information while rigorously developing algebraic number theory. It is suitable for undergraduates meeting the subject for the first time. Definitions are motivated and important concepts are illustrated by computational examples.

The material in the first 10 chapters is approximately equivalent to the Part II Number Fields course, landmarks being ideals, Minkowski's Theorem, and the class-group. The remaining chapters contain the proof of a special case of Fermat's Last Theorem (regular prime exponents), which uses all the previously introduced ideas. They also touch on more advanced topics leading up to a sketch proof of its general version.

The extra material on elliptic curves and elliptic functions has little to do with the rest of the book and feels a bit disconnected. However, the chapters on algebraic number theory are excellent for accompanying a university course, while the last part will whet the reader's appetite for more. *Philipp Kleppmann*

Movie Reviews

Genius, Insane, Blackmail, Murder…



A Beautiful Mind (2001) ★★★★

Russell Crowe, Ed Harris, Jennifer Connelly

This movie tells the story of John Forbes Nash (Russell Crowe): from Princeton University, where he searches for his *truly original idea*, to MIT, where he teaches calculus and marries one of his students Alicia (Jennifer Connelly), to winning the Nobel Prize.

When Nash is hired by the department of defence to find secret messages hidden in magazines and newspapers, and when he feels increasingly threatened by foreign agents, it becomes clear that he is schizophrenic and many parts of his life are nothing but hallucinations.

In between reality and imagination, genius and madness, love and reason, this quadruple Academy Award winning movie combines tension, great acting, fantastic music and a thrilling storyline. *PJL*

Good Will Hunting (1997) ★★★★

Robin Williams, Matt Damon, Ben Affleck

MIT professor and Fields Medallist Gerald Lambeau is stunned when he sees rebellious janitor Will Hunting (Matt Damon) solve the difficult maths problems on the hallway blackboard. When Will is sentenced to jail, Lambeau arranges for him to instead study mathematics and see a therapist (Robin Williams). While finding themselves, and what they really want, the two have to deal with expectations and defence, love, abuse and their past.

Full of emotions, passion and charisma, this motivating movie was nominated for nine Academy Awards, and won two.

PJL

Fermat's Room (2007) ★★★

Alejo Sauras, Ariadna Cabrol, San Yélamos

Four leading mathematicians are invited to spend a weekend solving one of the greatest enigmas of all times. With the door shut and the walls slowly moving closer, the real enigma is what connects them and why somebody wants to kill them…

The film starts with the words '*everybody who doesn't know what a prime number is should leave now*', though most of the puzzles presented are rather trivial. While the film doesn't quite live up to its Hitchcock models, the hidden hints and conversations make the film exciting and entertaining. Good actors, a fantastic set and great film music create a thrilling atmosphere.

Although only available in Spanish, it is easy to follow all conversations with English subtitles. *PJL*

21 (2008) ★★★

Jim Sturgess, Kate Bosworth, Kevin Spacey

Based on a true story: six gifted MIT students, led by the eccentric mathematics professor Micky (Kevin Spacey), spend their weekends in Las Vegas and use a complex set of signs and signals to beat Blackjack. However the glamorous (and highly profitable) life changes dramatically when both security chiefs and Micky demand their money.

References to Monty Hall and Newton-Raphson make the movie entertaining for mathematicians, though card counting doesn't require any advanced mathematical skills. The plot, however, is full of clichés and very predictable; it fails to build up tension or convey the thrill of gambling. *PJL*

Proof (2005) ★★

Gwyneth Paltrow, Anthony Hopkins, Hope Davis

Catherine (Gwyneth Paltrow) lives with and cares for her father Robert (Anthony Hopkins), a brilliant mathematician suffering from mental illness. After Robert's death, she is confronted with one of her father's students, who suspects the solution to a great mathematical problem in the old notebooks, as well as her sister (Hope Davis), who doubts Catherine's own mental state. Did she inherit both genius and madness from her father?

The movie, like many others, seems to suggest that scientific genius and madness are intrinsically linked – is there nothing else interesting about mathematics that is worth filming?

Though certainly worth watching, Proof doesn't convey tension, excitement, or any ideas and feelings to walk away with. *PJL*

Solutions

Another Fine Product from the Nonsense Factory…

1. Attitude Adjuster

Note that the raptor's acceleration is bounded, whilst yours is not. As is standard, we approximate the raptor and you as point particles. Hence run away on the you-raptor line at 6 m/s until it is close to you. Dodge sideways by epsilon at the appropriate moment. The Raptor must miss as its acceleration is bounded. Return to be directly behind the raptor by moving epsilon back again (higher order effects from the Raptor turning during this manoeuvre can be ignored for epsilon small enough.). Repeat this process. Since the Raptor ran more than 40 m to get to the point where you wiggle, it will run over 40 m before coming to a halt (for epsilon small). Hence this never terminates, and you can evade the raptor indefinitely.

2. Funny, It Worked Last Time…

Yes. The simulation showing this is rather too large for these margins.

3. What Are The Civilian Applications?

Sum of all entries in the table is $[0.5\, n\, (n+1)]^2 = 2n^3 + 4n^2 + 2n$. Solved by $n = 0, -1$ or 8. Hence Microraptors have 8 digits.

4. The Precise Nature of the Catastrophe

Write down the obvious matrix and find its eigenvectors; considering vectors as (Z, R, P):

$$v_1 = (0, -1, 1) \quad e_1 = 2, \qquad v_2 = (3, 2, 1) \quad e_1 = -1, \qquad v_3 = (-1, 0, 1) \quad e_3 = 1.$$

So the desired end state is proportional to $(6, 0, 0) = 2v_1 + v_2 - 3v_3$. log 3 days earlier, this would be $2/9\, v_1 + 3v_2 - v_3 = (10, 52/9, 20/9)$. Hence we wish to release $52/90\, P = 26/45\, P$ raptors.

5. Another Victim Of The Ambient Morality

0 questions.

1. Tell the PhD students that there is free food in Core.
2. Choose a door at random, and tell the law student to go through.
3. If the lawyer dies, you want the other room.

6. Me, I'm Counting

The number of even arrangements – the number of odd arrangements is:

$$\sum_{r=0}^{42}\binom{42}{r}2^{42-r}(-1)^r = \sum_{r=0}^{42}\binom{42}{n-r}(-2)^{42-r} = (1-2)^n = 1.$$

since n is even. Thus the number of even arrangements is greater than the number of odd ones.

7. Now We Try It My Way

Assume all at the north pole for convenience. Assume a human is 50 kg distributed over a 0.5 m × 0.25 m rectangle (ignoring height which is irrelevant). Assume 10 rad/s angular velocity (a little under 2 Hz)

Moment of inertia of one person = 60 kg × $(0.5^2 + 0.25^2)$ $\text{m}^2/3$ = 25 kgm^2.

Angular momentum of population = $6 \times 10^9 \times 25 \times 10 = 1.5 \times 10^{12}$ kgm^2/s.

Earth moment of inertia = $M_e \times 2/5\ R^2 \approx 8\pi / 15 \times 5000\ R^5 \approx 8000\ (6.4\ E^6)^5 = 2^{33} \times 10^{28} \approx 8 \times 10^{37}$ kgm^2.

So angular velocity down $\approx 5 \times 10^{-26}$ / s.

$\omega = 2\pi / 86400 = \pi / 43200 \approx 1/14400$.

Day length = $2\pi / (\omega - 5 \times 10^{-26}) = 86400 + 5 \times 10^{-26} \times 14400 \times 86400 \approx 86400 + 6 \times 10^{-17}$ s

So the answer is 6×10^{-17} s. Any approximations leading to an answer of the right vague order are good. (Results within 3 orders of magnitude were deemed acceptable.)

8. Experiencing A Significant Gravitas Shortfall

1. 10%: CMS Core,
2. 60%: B pavillion,
3. 30%: the INU,
4. 80%: the UL,
5. 0%: an ACME Klein Bottle.

This question was largely marked on the justifications given. Glass in the CMS was deemed to be far weaker in our analysis than in the general analysis of participants; many considered B pavillion to be the most secure, or felt that mathematicians could take on *raptors* in toe to claw fighting.

9. Just Read The Instructions

Another long simulation. The correct answer was: 1, from the bottom row, 3rd from the right. If you don't get this answer, then your raptors are insufficiently ingenious.

10. It's Character Forming

Elephant ≈ 5000 kg, Pool ≈ 2500 m^3, Blink ≈ 0.3 s.
Power = kgm/s^3 $\Rightarrow$ new unit is $\approx 5000 \times (2500)^{1/3} / 0.3^3 = 2500000$ W
Kettle ≈ 2kW so 1250 required.

11. Sleeper Service

The distance constraint implies that the raptors form a Hamming code on 7 bit sequences. Famously this allows for 16 raptors.

12. Ultimate Ship The Second

The vast majority of participants picked A. They should have used psychology.

This question has as open a mark scheme as Tripos.
Specifically, we looked at them all and decreed the winner.

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 Smith number, since the sum of its digits equals the sum of its prime factors excluding 1. 94! – 1 is prime.

Questionnaire

We would like to take this opportunity to ask for your opinion about Eureka, in particular the changes we have made to this issue. Please answer the questions below and return this form to the address on the back. Subscribers within the University of Cambridge can use the University Messenger Service (UMS) free of charge.

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