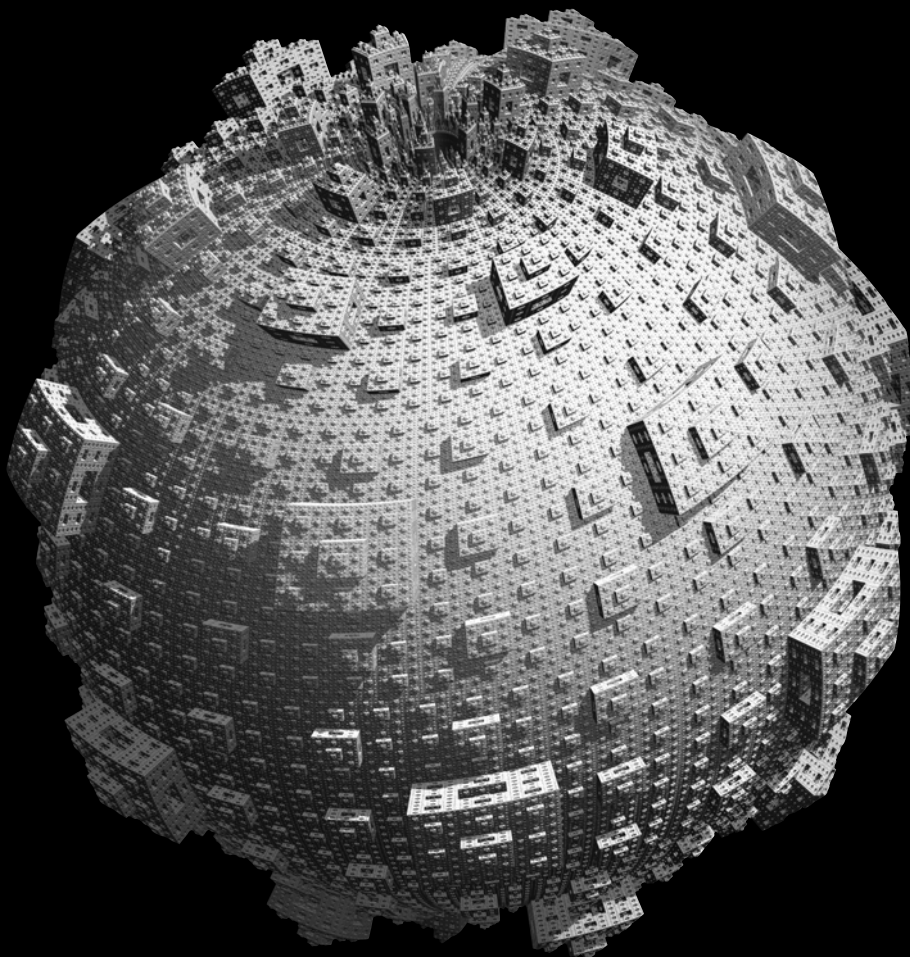


A JOURNAL OF THE ARCHIMEDEANS





The Menger Sponge World
a fractal created by Ramiro Perez

EUREKA 60 — A JOURNAL OF THE ARCHIMEDEANS

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Love and Tensor Algebra © 2009 Barbara and Tomasz Lem, Houghton Mifflin Harcourt Publishing

————— *More information about the journal and how to obtain it on page 80.* —————

EUREKA 60

A Journal of the Archimedean

Editors: Philipp Legner and Anja Komatar

60th Edition

Business Manager: Ben Spurgen

November 2010

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Foreword

by Ian Stewart

When I arrived at Churchill College in 1963, I had already heard of Eureka, thanks to the inimitable Martin Gardner, who had mentioned it in his celebrated Mathematical Games column in Scientific American. Martin died in 2010, and tributes poured in from the mathematical community. The most apt summary of his influence is a quote attributed to Ron Graham: “Gardner turned thousands of children into mathematicians, and thousands of mathematicians into children”. He did both to me, in that order.

Anyway, on arriving in Cambridge I hastened to join the Archimedians, and by a quirk of fate ended up editing Eureka. The previous editor, Peter Lee, was at Churchill, and also my Director of Studies: he took steps to keep it in the family. I found a note from him in my copy of issue 27, 1964, the 25th Jubilee issue. It reads: “And the best of British luck for you next time.”

Now, I may be biased, but I think Eureka is a brilliant magazine. Many of its articles are classics. In issue 13 (1950) Cedric A. B. Smith, under his pseudonym Blanche Descartes, answered the 12-ball weighing puzzle in a poem, whose crux was the lines

F AM NOT LICKED
MA DO LIKE
ME TO FIND
FAKE COIN

Exercise for the reader: work out what the devil I’m rabbiting on about.

I also remember vividly the article “Train Sets” by Adam Chalcraft and Michael Greene (issue 53, 1994), which interpreted the Halting Problem for Turing machines in terms of the layout of a toy railway track. Conclusion: there is no algorithm to predict, for any layout, whether the train will eventually reach the station. I’m pretty sure this connection can be turned into a dynamical system with undecidable dynamics, and may even write this up some time. It nicely complements fundamental work on this question by Christopher Moore of the Santa Fe Institute.

When I left Cambridge to do a PhD at Warwick, my experience with Eureka helped when a group of us founded our own mathematical fanzine, Manifold, which

Gardner also mentioned occasionally. Through a series of coincidences too tortuous to relate here, I ended up becoming the fourth person to write Gardner's column, now renamed Mathematical Recreations. So I have the Archimedeans, and Eureka in particular, to thank for my involvement in the popularisation of mathematics.

Manifold folded after 20 issues and 12 years. Eureka has greater longevity, and is still going strong, with 60 issues over 71 years (the first issue was in 1939). I'm sure that by 2050 or earlier it will have reach its 100th issue, though by then it will probably exist in some version of William Gibson's cyberspace, not on that funny stuff they used to call "paper", or even in a (shock, horror) machine.

It is a privilege and a pleasure to celebrate the 60th issue of Eureka, and to thank all of the editors, over the last six decades, for making the mathematical world a more pleasant and interesting place. To quote the ancient Egyptian phrase:



Life, Prosperity, Health!

Oh let us raise a foaming beaker,
Of Nescafe in cup styrene,
To praise the 60th Eureka,
No finer journal can be seen,
Nor greater pleasure does my post afford,
Than sending blessings from the Faculty Board.

*Professor Tom Körner,
Chairman of the Faculty Board
Faculty of Mathematics*

Editorial

60 Issues of EUREKA

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The number 60 has an important meaning in many aspects of life: there are 60 minutes in one hour, 60° in an equilateral triangle and 2×6 months in a year. This is a legacy of the Babylonian number system, which first appeared around 3500 BC. As with Roman numerals, all numbers from 1 to 59 can be represented by only two symbols 𐎶(1) and 𐎵(10); an essential improvement over previous number systems requiring a different symbol for each number. The Babylonians were also the first to use a positional number system: all numbers bigger than 59 have multiple digits and decimal digits were written as multiples of $1/60$. Base 60 was chosen because of the large number of divisors: 60 is the smallest number divisible by all integers from 1 to 6.

60 was also important in many other cultures: the Chinese calendar, for example, is based on the *Sexagenary Cycle* (六十花甲). This cycle arises since 60 is the lowest common multiple of 10 and 12. Traditionally, the Chinese count years using a 10-year cycle, called the 10 Heavenly Streams (consisting of ying, yang combined with the five elements earth, water, wood, fire) and using a 12-year cycle, called the 12 Earthly Branches (consisting of the 12 animal signs). 2010 is the year of the tiger, yang and metal – the current sexagenary cycle will finish in 2043.



In Christian culture, 6 has a rather dark meaning and 666 is even called the “*Number of the Beast*”. The number 666 appears several times in the Bible, and in other old scripts. The ancient writers probably didn’t know that 666 is related to the golden ratio by $\phi = -2 \sin(666^\circ) = -2 \cos(6 \times 6 \times 6^\circ)$, but maybe they were aware of some of the following “scary” relationships:

$$\begin{aligned} \varphi(666) &= 6 \times 6 \times 6, & \varphi \text{ is Eulers Totient Function} \\ 666 &= 3^6 - 2^6 + 1^6, \\ 666 &= 1 + 2 + 3 + \dots + 36, \\ 666 &= 2^2 + 3^2 + 5^2 + 7^2 + 11^2 + 13^2 + 17^2, \end{aligned}$$

$$666 = 1 + 2 + 3 + 4 + 567 + 89 = 123 + 456 + 78 + 9.$$

In addition, it is possible that people were fascinated by 666 since in roman numerals it can be written as DCLXVI, using all letters other than M.

Do you think it is an accident that there are 6 types of quarks and 6 types of leptons in the standard model of particle physics? Or that the smallest non-abelian group has order 6 and the smallest non-abelian simple group has order 60? It is even said that there are *6 degrees of separation* between any two human beings on earth!

Hopefully those disturbing facts about 6 don't prevent you from reading this journal, which includes many interesting articles about mathematical topics, as well as problems, book reviews, art, poems and humour. We are delighted to publish articles by Ian Stewart, who was editor of Eureka while studying at Cambridge, Clifford Pickover, well known science (and science fiction) author and columnist, and David Tong, fellow of Trinity College. In addition, many students have submitted articles about a variety of interesting and exciting topics.

Over the past 60 issues, Eureka has developed from a magazine read by mathematicians in Cambridge to a journal that is known all over the world. During the last year we received letters and emails from India, Iran, South Korea, Germany, Australia, Saudi Arabia, the United States and many other locations – unfortunately we couldn't publish all of them!

We very much hope that you enjoy reading this jubilee-issue of Eureka.

*Philipp Legner,
St John's College*

Acknowledgements

The publication of this journal would not have been possible without the help and support of many students and professionals. I especially want to thank Michel Atkins for proofreading the journal and his advice on style and layout. I would also like to thank the Archimedean committee, who gave us useful feedback and support. Roddy MacLean and Noel Robson from Cambridge University Press kindly advised us on the layout and printing of the journal. Many thanks are also due to Barbara and Tomasz Lem, and to Houghton Mifflin Harcourt Publishing for giving us the permission to publish the poem on page 49.

Finally, I would like to thank everybody who contributed to the journal – by writing articles, granting us permission to use images or in any other way. The continuous success of Eureka depends on your support and effort.



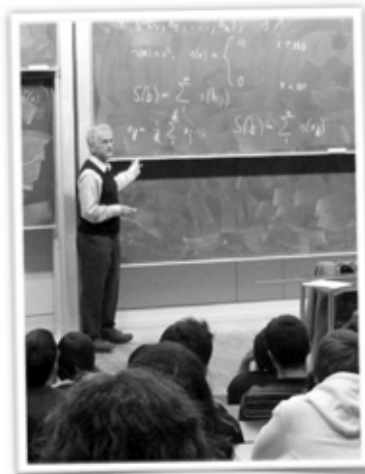
The Archimedean's
The Cambridge University Mathematical Society

This term has been one of the most successful in the history of the Archimedean's. The committee was very busy during the summer, designing a new logo and a new website which has proved popular with members. The Freshers Fair at the start of term has been our best ever, with the '1 minute challenge' and free giveaways bringing in over 150 new members and many more joining our mailing list.



Our lectures this year are focussing on the Millennium Prize Problems; the first of a series of talks was given by Prof. Ben Green on the Riemann Hypothesis. The talk was followed by our traditional Freshers' Squash, and the provision of pizza seem to have attracted twice the usual number of students.

The other highlight of the term was Prof. Bollobás' talk on 'Cambridge Gems' — famous theorems proved by Cambridge Mathematicians, from Hardy and Littlewood to Erdős. We even had to move to a bigger lecture theatre because it was so well attended. After the talk, Prof. Bollobás and his wife generously invited all of us to their house for cheese and drinks. Needless to say, we are indebted to them for their hospitality in hosting such a wonderful party.





The end of Michaelmas Term is coming closer and closer, and we are looking forward to many interesting talks this and next term, as well as our triennial dinner.

Being on the committee has been a fun and ultimately rewarding experience. If you are interested in getting involved, either now or in the future, please get in touch. It only remains for me to thank the current committee for the efforts, and the members and subscribers to Eureka for their continued support.

*Lovkush Agarwal,
President 2010-2011*

The Committees 2009–2010 and 2010–2011

President:	Tom Ducat (Fitzwilliam)	Lovkush Agarwal (Corpus Christi)
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Registrar:	Urs Schoenenberger (Fitzwilliam)	
Publicity:		Fangzhou Liu (Sidney Sussex)
Webmaster:		Zhu Gong (Lucy Cavendish)

Calendar of Events

Michaelmas Term 2010

14 October	The Riemann Hypothesis – a talk by Professor Ben Green (DPMMS) <i>Find out more about the most famous unsolved problem in Maths.</i>
5 November	Cambridge Gems – a talk by Professor Bela Bollobas (DPMMS) <i>A rare opportunity to hear the winner of the Senior Whitehead Prize, holder of an 'Erdos Number' of 1 and simply one of the best speakers in Mathematics. The talk is about three results by some of the most eminent Cambridge mathematicians of the last century.</i>
12 November	Tuning with Turing – a talk by Professor Andrew Thomason (DPMMS) <i>What is a Turing machine? And is a Turing Machine smarter than a 10 year old? We shall try to answer these questions and to offer a brief panorama of the mathematics behind computers.</i>
19 November	Yang-Mills Theory – a talk by Professor David Tong (DAMTP) <i>YMT used geometrical structures to describe elementary particles, but there are still some problems regarding the "mass gap".</i>
TBC	Annual Dinner

Lent Term 2011

21 January	Diophantine Equations – a talk by Professor Samir Siksek (Warwick) <i>In his unique and appealing style, Siksek will convey his own joy for this intriguing area of mathematics which has been studied for millennia.</i>
4 February	The Hodge Conjecture – a talk by Professor Richard Thomas (Imperial) <i>Discover how the areas of algebra and geometry are deeply connected.</i>
25 February	P vs NP Complete – a talk by Professor Anuj Dawar (Computer Lab) <i>The practical implications of this problem affect all areas of modern life, from data encryption to curing cancer!</i>
TBC	The Poincaré Conjecture – a talk by Simon Donaldson (Imperial) <i>Who can tell you more about the only solved Millennium problem than a member of the advisory boards of the Clay Foundation.</i>
TBC	Navier-Stokes Equation – a talk by Professor Hubert Huppert (DAMTP) <i>Find out about these equations, whose deep secrets will lead to the understanding of all 'wakes' of phenomenon in fluid dynamics.</i>
TBC	Annual General Meeting

Summer Term 2011

TBC	Science Societies Garden Party
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Please visit our website <http://www.archim.org.uk/> for information about locations and times, as well as events such as the **Fresher's Squash**, **Improvisational Comedy Ents** or the **Annual Problems Drive**.

The Archimedean

Problems Drives 2009 and 2010

Question 1 (2009)

A country consisting of 100 states, each having a population of 100, is having a presidential election, with two candidates running. The election runs in two stages – first by state and then nationwide – and the rules are below:

State voting:

- If fewer than 75 votes are cast, the state doesn't vote in the nationwide election.
- If at least 75 votes are cast, the state casts a single vote for the candidate with the majority of votes.

Nationwide voting:

- If fewer than 75 states cast votes, the election is invalid (and anarchy ensues).
- If at least 75 states cast votes, the candidate with the majority wins.

Let A be the least percentage of the population that could vote for a candidate who goes on to win and B the greatest percentage of the population that could vote for a candidate who does not go on to win. What is $A - B$?

Question 2 (2009)

Find the maximum value of

$$f(x) = \sqrt{x^4 - 5x^2 - 8x + 25} - \sqrt{x^4 - 3x^2 + 4}$$

Question 3 (2009)

Fill in the two blanks of each of the following sequences:

- (a) 3, 5, 11, 15, 21, __, __
- (b) __, 12, 6, 3, 5, 4, __
- (c) 0, 1, 8, 81, 1024, __, __
- (d) 1, 2, 4, 7, 8, __, __
- (e) 4, 6, 12, 18, 30, __, __

Question 4 (2009)

If a positive number has an even number of 1's in its binary expression, then it's said to be a magic number. Find the sum of the first 2008 magic numbers (as a decimal).

Question 6 (2009)

You walk through the apple orchard illustrated below, where the numbers represent the number of apples in each tree. You start at **A** at and must always move to the right at each step, but may choose to stay in the same row, or move one up or one down. You must also end at **B**. Furthermore, every time you move between columns you have a 50% chance of being caught by the farmer, in which case you will lose all the apples you are carrying, but will be allowed to continue (don't ask why!). If you choose an optimal route through the orchard, what is the expected number of apples you will have at **B**?

	1	5	0	6	1	4	
	3	6	9	1	2	5	
	4	9	8	2	0	2	
A	5	1	4	3	4	5	B
	3	3	2	1	9	2	
	2	2	5	7	3	7	
	9	1	2	8	9	8	

Question 7 (2009)

Find the integer part of

$$x = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{1010025}}$$

(Hint: $\sqrt{1010025} = 1005$)

Question 8 (2009)

We have nine balls of the same size in a box, labelled 1, 2, 3, ..., 9. Richard picks up a ball randomly, labelled with the number x , and puts the ball back in the box. Ed then takes out a ball, which has number y on it. Find the probability that

$$x - 2y + 10 > 0.$$

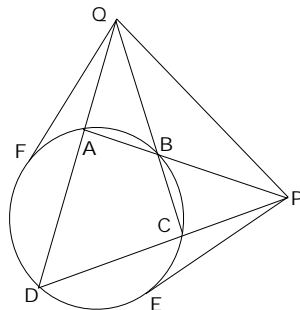
Question 9

Match the mathematician to his book:

Carl Friedrich Gauss	Vollständige Anleitung zur Algebra
Isaac Newton	Explication de l'Arithmétique Binaire
Leonhard Euler	Arithmetica Universalis
Daniel Bernoulli	Hydrodynamique
Gottfried Leibniz	Disquisitiones Arithmeticae

Question 10 (2009)

A , B , C and D are four points on a given circle (clockwise), and $AD \cap BC = Q$ and $AB \cap DC = P$ (i.e. AD and BC intersect at point Q outside the circle, while AB and DC intersect at point P outside the circle). Moreover, E and F are two points on the circle such that PE and QF are tangent to the circle. Now, given that $|QF| = 17$ and $|PE| = 19$, find $|PQ|$ (here $|QF|$ denotes the distance from Q to F , etc).



Question 12 (2009)

Choose an integer $0 \leq n \leq 3$. If the number of teams choosing your number is congruent to $n \pmod{4}$ then you will get $\frac{1}{n+1}$ points, otherwise you will get nothing.



Question 2 (2010)

An economical carpenter had a block of wood measuring eight inches long by four inches wide by three and three-quarter inches deep. How many pieces, each measuring two and a half inches by one inch and a half by one inch and a quarter, could he cut out of it? (Please offer an explanation of the cutting process.)

Question 3 (2010)

Please arrange the following numbers in increasing order (years are to be interpreted in the Gregorian calendar):

- A the magic constant of a 16×16 normal magic square
- B the number of seconds in a millifortnight
- C the smallest number expressible as the sum of two positive cubes in two different ways
- D the year of the foundation of Archimedean
- E the year of Isaac Newton's death
- F (rest mass of a proton) / (rest mass of an electron)
- G great gross

Question 4 (2010)

Smith, Brown and Jones agree to fight a three-way pistol duel. After drawing lots to determine who fires first, second, and third, they take their places at the corners of an equilateral triangle. It is agreed that they will fire single shots in turn and continue in the same cyclic order until two of them are dead. At each turn the man who

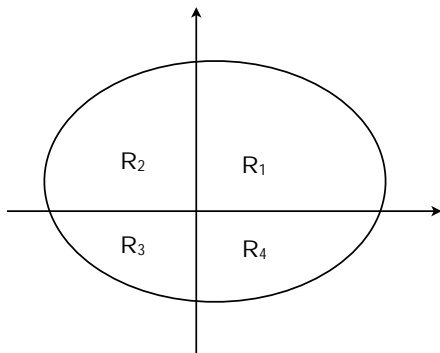
is firing may aim wherever he pleases. It is well-known that Smith is a perfect shot, Brown is 80% accurate, and Jones hits his target precisely half the time. Assuming that all three adopt the best strategy, and that nobody is killed by a wild shot not intended for him, determine the survival chances of each of the three duelists.

Question 5 (2010)

In the diagram (not to scale) you see the ellipse described by the equation

$$\frac{(x-20)^2}{20} + \frac{(x-10)^2}{10} = 2010.$$

Let R_i denote the four areas bounded by the ellipse that are in the first, second, third, and fourth quadrants respectively. Find $R_1 - R_2 + R_3 - R_4$.



Question 6 (2010)

Each of the 9 squares in this diagram contains a digit between 1 and 9, all the digits are distinct, and all the equations are satisfied. Fill in the square.

$$\begin{array}{rcccl} \square & - & \square & = & \square \\ & & & & \times \\ \square & / & \square & = & \square \\ & & & & = \\ \square & + & \square & = & \square \end{array}$$

Question 7 (2010)

In a three-by-three matrix of squares, an ordinary six-sided die is placed in the centre square. On a move you may roll the die to an adjacent cell. The cells of the matrix are labelled (1,1) through (3,3); the die is therefore in cell (2,2). The die is oriented so that 1 is on top, rolling it into cell (2,1) will place 2 on top, and rolling it into cell (3,2) will place 3 on top. Roll the die into cell (3,1) so that 6 is on top using as few moves as possible.

Question 10 (2010)

There are 8436 steel balls, each with radius 1 centimetre, stacked in a tetrahedral pile, with one ball on top, 3 balls in the second layer, 6 in the third layer, 10 in the fourth, and so on. Determine the height of the pile in centimetres.

Question 8 (2010)

Find the next two members of the following sequences:

- (1) 1, 1, 4, 9, 25, 64, __, __, ...
- (2) 1, 11, 21, 1211, 111221, __, __, ...
- (3) 1, 2, 4, 7, 28, 33, 198, __, __, ...
- (4) 3, 3, 5, 4, 4, 3, __, __, ...
- (5) 1, 2, 5, 10, 20, __, __, ...
- (6) 1, 11, 4, __, __

Question 11 (2010)

Below are three lists, containing a work, its publication date, and its author. Match the correct triples.

1202	Carl Friedrich Gauss	The Mathematical Analysis of Logic
1644	René Descartes	Book of the Abacus
1705	Augustin-Louis Cauchy	Principia philosophiae
1724	Abraham De Moivre	An Historical Account of Two Notable Corruptions of Scripture
1730	Leonardo Fibonacci	Synopsis Astronomia Cometicae
1754	George Boole	The Differential Method
1801	James Stirling	Annuities on Lives
1823	Edmond Halley	Disquisitiones Arithmeticae
1847	Sir Isaac Newton	Le Calcul infinitésimal

Question 12 (2010)

Write the rank in which you think your team is with the scores up to, but not including, this question. A correct answer is worth 1 point; an incorrect, 0.



The 2010 Problems Drive was written by Michael Donaghy and Elena Yudovina.

Hints and solutions to the questions can be found on page 76.

The full problems drive is published on our website.

9. What is the solution of the following equation: $3x - 2 = 2x + 1$?
- (a) -1 (b) $\frac{3}{5}$ (c) 3 (d) $-\frac{1}{5}$
10. What is the next number in the following sequence: 2,5,10,17,26,...?
- (a) 37 (b) 35 (c) 41 (d) 42
11. $\int_{-\infty}^{\infty} xe^{-e^{-x^2}} dx = ?$
- (a) $\sqrt{\pi}$ (b) 1 (c) ∞ (d) 0
12. What is the length of the side marked with x in the right-angled triangle?
- (a) $\sqrt{7}$ (b) 5 (c) $\sqrt{6}$ (d) $\sqrt{8}$
13. $\frac{(3!)!}{(3!)^2} = ?$
- (a) $\frac{2}{3}$ (b) 1 (c) 40 (d) 20
14. Differentiate e^{e^x} with respect to x .
- (a) e^{e^x} (b) $e^x e^{e^x}$ (c) $xe^x e^{e^x}$ (d) $e^{e^{e^x}}$
15. What are the solutions of $2x^2 - 3x - 9 = 0$?
- (a) $3, -\frac{2}{3}$ (b) $3, \frac{2}{3}$ (c) $-3, \frac{2}{3}$ (d) $-3, -\frac{2}{3}$
16. Which of the following numbers is the largest?
- (a) 6 (b) e^2 (c) $4\sqrt{2}$ (d) $\frac{2}{3}\pi^2$
17. $\sqrt{2} \left(\cos^2\left(\frac{\pi}{6}\right) - \frac{1}{2} \right) = ?$
- (a) $\sqrt{2}$ (b) $\frac{\pi}{\sqrt{2}}$ (c) $\frac{1}{2\sqrt{2}}$ (d) $-\frac{\sqrt{2}}{4}$
18. How many rational roots does $\sin(\sqrt{2}\pi x)$ have?
- (a) 1 (b) 0 (c) 2 (d) ∞
19. How many 3-digit numbers are there with only even digits?
- (a) 125 (b) 450 (c) 451 (d) 100
20. What is the right answer to this question?
- (a) a (b) c (c) d (d) b



The fundamental Laws of the Universe?

Physics and the Integers

Dr David Tong, DAMTP

Leopold Kronecker famously said: “God made the integers, all else is the work of man.” In the context of late 1800s mathematics, this was a controversial viewpoint; a polemic against developments such as irrational numbers, Cantor’s set theory and the Bolzano-Weierstrass theorem. It did not make Kronecker a popular man.

More than a century later, no mathematician would deny the importance and utility of the developments that Kronecker railed against. Yet I suspect that many harbour some sympathy for his statement. The integers hold a special place in the heart of mathematics. Many of the most famous unsolved conjectures relate to the properties of the primes. More importantly, the integers are where we start mathematics: they are how we count.

In this essay I would like to view Kronecker’s quote through the lens of theoretical physics. Tested against our best theories of Nature, I will argue that the statement is wrong. Experimentally, falsifiably, wrong.

It is not obvious that the integers have any place in physics. The counting that is evident in mathematics is not so easy in the real world. I was told at school that there are 9 planets in the solar system. Now there are 8. Or maybe 13. As this example shows, the problem of finding the integers in Nature lies not in the counting, but rather in the defining. The Kuiper belt contains objects ranging in size from a few thousand kilometers to a few microns. You can only decide which objects are planets and which are merely lumps of rock if you employ a fairly arbitrary definition of what it means to be a planet. To find the integers in physics, we need Nature to provide us with objects which are naturally discrete.

Fortunately, such objects exist. While the definition of a planet may be arbitrary, the definition of an atom, or an elementary particle, is not. Historically, the first place that the integers appeared was in the periodic table of elements. The integers labelling atoms – which, we now know, count the number of protons – are honest. Regardless of what developments occur in physics, I am sure that we will never observe a stable element with $\sqrt{500}$ protons that sits between titanium and vanadium. The integers in atomic physics are here to stay.

In fact, once we are in the atomic world, the integers are everywhere. This is what the “quantum” of quantum mechanics means. For example, you learn in the second year course that the spectrum of energy levels of hydrogen are given by $E = -E_0/n^2$ where E_0 is the ground state energy and $n \in \mathbb{Z}$. More subtle quantum effects can even coax the integers to appear in macroscopic systems: the *quantum Hall effect* is a phenomenon that occurs in semi-conductors that are placed in a magnetic field. The Hall conductivity, which describes how current flows perpendicular to an applied electric field, is given by $\sigma = n(e^2/h)$ where e is the electron charge and h is Planck's constant and, once again, $n \in \mathbb{Z}$. These integers have been measured to an accuracy of one part in a billion, one of the most precise experiments in all of physics.

While the integers undoubtedly arise in physics, they do not have the same status of building blocks that they do in pure mathematics. They are not inputs of quantum theory, they are outputs. There are no integers in the Schrödinger equation describing the electron orbiting a proton. The fact that the solutions and the energy levels are labeled by integers is due to a normalisation condition, imposed so that the wave-function has a physical interpretation. Expressed more mathematically, the integers arise from the eigenvalue problem for continuous Hermitian operators. In physics, the integers are an example of an emergent quantity, no more fundamental than the concepts of temperature, smell, or the offside rule.

Perhaps more surprisingly, the existence of atoms — or, indeed, of any elementary particle — is also not an input of our theories. Despite what you are told in high school and popular physics books, Democritus was wrong. The basic building blocks of Nature are not discrete particles, such as the electron or quark. Instead our fundamental laws of physics describe the behaviour of *fields*, continuous fluid-like objects spread throughout space. The electric and magnetic fields are familiar examples, but our best description of reality adds to these an electron field, a quark field, and several more. The objects that we call fundamental particles are not fundamental. Instead they are ripples of continuous fields, tied into apparently discrete lumps of energy by the framework of quantum mechanics¹. In this way, the discreteness of the atomic world emerges². The framework which describes how the fields move is called *quantum field theory*. The specific quantum field theory that explains our world is the crowning glory of 400 years of scientific investigation. Unfortunately it has a rubbish name: it is called the Standard Model.

So much for the integers in the known laws of physics. But what about the laws of physics that we have yet to discover? The Standard Model is certainly not the last word and it is a common speculation that when we understand Nature on some deeper level it will turn out to be based on discrete mathematics such as the integers. Such speculation often comes from computer scientists who envision that the laws of physics will, at heart, be reduced to something akin to a computer algorithm. Is this

likely? Of course, no one knows. Here I would like to draw attention to an important open problem in the Standard Model that is little discussed but which may have bearing on the issue.

Before trying to find new laws of physics that are discrete, it would seem sensible to attempt to write down the known laws of physics in a manner which is at least compatible with a discrete underlying structure. Such a formulation is not just of academic value. The equations underlying quantum field theory are hard and humans are not very good at solving hard equations. Computers are much better. To formulate the laws of physics in a discrete manner means to write them in such a way that they could be simulated on a computer. As you may imagine, this is important in all areas of physics and a great deal of effort has gone into it. It is therefore rather surprising to learn that no one knows how to formulate a discrete version of the Standard Model.

At first sight, this seems very strange. All the laws of physics that you learn as an undergraduate are formulated in terms of differential equations and, while it may be difficult in practice to get reliable numerical results for certain partial differential equations, there is no problem of principle. One simply needs to replace continuous derivatives with finite differences. However, the mathematics that underlies quantum field theory is not the differential equations of classical physics, but instead an object known as the *path integral*. First introduced by Feynman, this is a functional integration – meaning that one doesn't integrate over a domain in, say, the real numbers, but instead over a domain of functions. This means that one must perform an infinite number of integrations.

For physicists the path integral has been the weapon of choice for half a century. The intuition gained in learning to manipulate these objects has led to many of the most beautiful results in theoretical physics. Yet mathematicians have to date been unable to make sense of the path integral in all but simplest cases. The infinities that arise in performing functional integrations, long since understood by physicists, are beyond the limits of rigorous analysis. Of course, that hasn't stopped physicists using the path integral to great effect³ and one of the most important tools that has been developed is a discretised version of quantum field theory which can be simulated on a computer. Usually called lattice field theory, the numerical evaluation of the path integral is performed using Monte Carlo techniques.

However, there is one class of quantum field theories that physicists do not know how to simulate on a computer. This is the class that involves particles called “chiral fermions”. To fully describe what a chiral fermion is would, unfortunately, take too long⁴, but they are wonderfully subtle objects that owe their existence to several intricate mathematical facts about the structure of space-time and the nature of forces

in quantum field theory. Most importantly for our story, all the particles in Nature, described by the Standard Model, are chiral fermions. (Although, peculiarly, only the weak nuclear force and the Higgs boson notice this fact.)

It is not entirely clear what to make of our inability to simulate the Standard Model on a computer. Perhaps this is merely a difficult problem waiting to be solved with conventional techniques. But it smells deeper than that. The obstacles that lie behind attempts to discretise chiral fermions are related to aspects of geometry, topology, index theorems and a physics version of Hilbert's Hotel known as the quantum anomaly. All of these rely on the continuous nature of the field. It may well be that our failure is telling us something important: the laws of physics are not, at heart, discrete. We are not living inside a computer algorithm. Probably.

Footnotes

- 1: A heuristic and simple explanation of why fields, rather than particles, underlie the world can be found in the introductory section of the quantum field theory lecture notes available for download at <http://www.damtp.cam.ac.uk/user/tong/qft.html>.
- 2: It might appear that the integers arise in a meta-fashion in physics since it looks as if the number of species of particle must be an integer. After all, one might think that we can count them: there is an electron field, a neutrino field, six quark fields, and so on. But this is illusory. There are interactions. One type particle can morph into other types and the boundary between them becomes blurred. It is still an open mathematical problem to understand how to count the number of particle species in a given quantum field theory. This problem has been solved only in a certain special classes of quantum field theory known as *conformal field theories* where the number has the rather technical name of 'central charge'. In general, it is not an integer.
- 3: Ironically, the path integral has also yielded some of the most interesting results in geometry in the past decades. Witten's Fields medal winning work on knot invariants was derived from the path integral approach to quantum field theory, as are other breakthrough ideas such as mirror symmetry and Seiberg-Witten invariants.
- 4: Here is a short description of a chiral fermion. All elementary particles carry a property called spin. For our purposes, it will suffice to think of the particles as tiny spinning balls. A "fermion" — named after the physicist Enrico Fermi — tells us that these particles carry $1/2$ unit of spin, the smallest amount any particle can carry. This already has strange consequences. If you rotate the particle by 360° , it doesn't come back in the same state. You need to rotate by 720° to achieve this. Now onto the word "chiral". This is a property only of massless fermions. Massless particles are restless, they keep moving, always at the speed of light. The word "chiral" means that the spin of the particle always rotates in a fixed direction, either clockwise or anticlockwise, around the direction of motion. One of the great shocks of the Standard Model of particle physics is that all the particles that we know — the electron, the quarks, the neutrino — are massless. They want to travel at the speed of light. The reason that they do not is the Higgs field, a treacle-like substance spread throughout space through which all other particles have to plough their way.

Sand Ripple Dynamics

Granular Flows and Washboard Roads

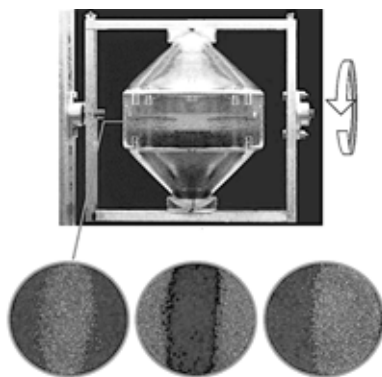
Peter Hatfield, Pembroke College

As a first year undergraduate, I spent comparatively little time at CMS – and no time at all in the laboratories! It was therefore a great pleasure to spend the summer after my first year working in DAMTP's G.K. Bachelor Laboratory with the Granular Flows group, mainly with Dr Jim McElwaine (Fellow at St. Catherines) and Dr Nathalie Vriend (Post-doctoral researcher).

Granular flows are the subdivision of Fluid Dynamics that deals with the flow of particles that are sufficiently large that they are not subject to thermal motion fluctuations and cannot be modelled with continuum mechanics. Interestingly the UK is unique in including fluid dynamics in mathematics departments – in Europe and the US the field is variously described as physics or engineering.

Granular flows (as I have learned!) can be very interesting, because they exhibit a wide variety of behaviours, from like a liquid to like a solid – and when these behaviours clash very complex behaviour can arise indeed. There are important implications: some estimates suggest 10% of the world's energy is used processing granular materials – think of sand, snow, grain, pills and cement.

A simple example to illustrate the complex behaviour that can arise is an experiment mixing equal volumes of two types of sand of different particle size. One is coloured red, the other blue. The sand is initially uniformly mixed. When the container is spun below a certain frequency, the larger particles will separate out and go to the side, while the smaller particles will concentrate in the centre. Interestingly though, if it is spun at a higher frequency, the opposite happens; the coarser particles will congregate in the centre. How-



Nonequilibrium Patterns in Granular Mixing and Segregation, Troy Shinbrot and Fernando Muzzio, *Physics Today*

ever, if the container is spun at exactly the threshold frequency, neither will happen – the symmetry of the system will break and the finer grains will go to the left and the coarse grains to the right (or vice versa). Just one of the unusual macroscopic effects in granular dynamics that is not really understood at all, even though at a particle level it is completely understood – simply Newtonian dynamics.

A current area of research in the department is how sand flows down gulleys, for example in avalanches. Peter Saunders (3rd year Physics at Downing) worked over the summer in the same lab with me looking at static areas that can form as the sand initially moves down the slope, how the curvature of the flow can change with flow rate and a variety of other phenomena.

Faraday heaping, a phenomenon originally observed by the man himself back in 1831, was only really properly sorted out in the last 5 years or so, a lot of work done by a group at the University of Twente, in the Netherlands. Essentially, when a flat bed of sand is shaken vertically, small heaps/hills will form almost immediately. Then, over long time periods, these heaps will join together to form one larger one. Crucially however, this does not happen when the grains are a vacuum, meaning the drag from the air in the cycle of an individual grains motion is key.

“Washboarding”, the effect by which the ripples you occasionally see in roads form, as well as moguls in sand etc. The same process is also at work in CD readers, where the reader passes over the CD continually and can cause deformations. Washboarding can often lead to roads having to be repaved, and can be a serious problem in parts of the developing world. One of the first to study wide-scale motion of sand was a certain Brigadier Ralph Bagnold OBE FRS. He was a Commander in north Africa during the Second World War, and also found time to write “The Physics of Blown Sand and Desert Dunes” (1941). Still an influential text, it was used by NASA recently to study the movement of dunes on Mars. The Bagnold Formula is named after him, relating wind speed and sand movement.

Segregation (which is what I mainly looked at over the summer) is the effect by which two different particle types will separate out when shaken together. This can be driven by both differences in particle size and particle density. There is an easy demonstration you can try in your kitchen to show a similar effect. Get a see-through jar or similar. Put a bolt or something of similar size and weight in, and also something of similar size, but much lighter like a peg. Mix it around a bit and then try shaking it vertically. You should find the bolt rises to the top, and the peg sinks to the bottom. Now put the jar on its side and shake horizontally. You should find the opposite happens! You have just demonstrated the “Brazil-Nut effect” where Brazil nuts rise to the top of mixed nuts. The vertical effect is understood to an extent – the horizontal effect apparently remains a mystery...

Overall I found granular flows to be an intriguing area of active research. A very interesting summer - I'm sure there remain many more things to be discovered in the field in the future that will both fascinate mathematicians and be of importance to physics and engineering.

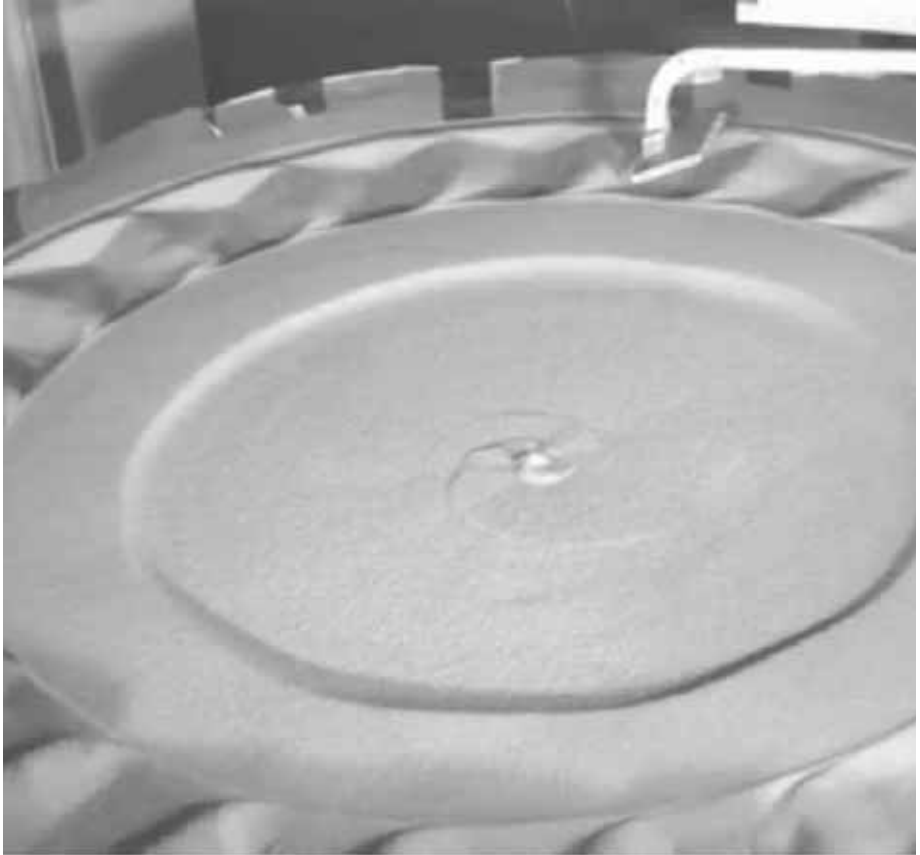


Image by Jim McElwaine

Factorimal Expansions and the Irrationality Machine

Prof. Ian Stewart, University of Warwick

Introduction

One of the most prominent special numbers in mathematics is e . Joseph Fourier [3] proved that e is irrational, and Charles Hermite proved it is transcendental; that is, it satisfies no nontrivial polynomial equation over the rationals.

Here we reinterpret the irrationality of e using some simple ideas from dynamical systems and a special type of multibase expansion. It is well known that standard decimal notation can be modified to use any integer greater than 1 as a base, not just 10. In multibase systems the base can change according to the digit concerned. For example, measurement in pounds and ounces employs base 16 for ounces but base 10 for pounds, and the old pounds-shillings-pence system for British money used base 10 for pounds, 20 for shillings, and 12 for pence.

The irrationality proof uses successive bases 2, 3, 4, 5, ... to the *right* of the analogue of the decimal point. Since this notation creates factorials in the algebra, we call it the ‘factorimal’ expansion.

The Factorimal System

A *factorimal* is an expansion of the form

$$x = \sum_{n=2}^{\infty} \frac{\theta_n}{n!} \quad (2.1)$$

where $\theta_n \in \mathbb{N}$ and $0 \leq \theta_n < n$. Any such series converges absolutely.

Inductively, it is easy to prove that the finite series $x = \sum_{n=2}^k \frac{\theta_n}{n!}$ are precisely the rational numbers of the form $\frac{m}{k!}$ where $0 \leq m < k!$. This means that factorimals are not plagued by infinite recurring expansions for rationals, like 0.33333... for $1/3$ in base 10. More precisely:

Theorem 2.1: Every terminating expansion represents a rational number in $[0,1)$, and every rational number in $[0,1)$ has a terminating expansion.

Proof: The first statement is obvious. For the second, consider a rational $0 \leq p/q < 1$. We can write $p/q = r/s!$ where r is not divisible by s (take the smallest possible such s). We prove inductively on s that $r/s!$ has an expansion stopping at $\theta_s/s!$.

When $s = 2$ the possibilities are $0/2, 1/2$ and the result is obvious.

When $s > 2$, use the division algorithm to write $r = qs + t$ where $0 \leq t < s$. Then

$$\frac{p}{q} = \frac{r}{s!} = \frac{qs + t}{s!} = \frac{q}{(s-1)!} + \frac{t}{s!}$$

By induction,

$$\frac{q}{(s-1)!} = \sum_{n=2}^{s-1} \frac{\theta_n}{n!}$$

If we set $\theta_s = t$ then

$$\frac{p}{q} = \sum_{n=2}^s \frac{\theta_n}{n!}$$

□

Since these numbers are dense in $[0,1)$ the following theorem is no surprise:

Theorem 2.2: Every $x \in [0,1)$ has a factorimal expansion, which may be infinite.

Proof: Given x , define the θ_n inductively by

$$\theta_2 = \lfloor 2x \rfloor \tag{2.2}$$

$$\theta_k = \max \left\{ \theta : \sum_{n=2}^{k-1} \frac{\theta_n}{n!} + \frac{\theta}{k!} \leq x \right\} \tag{2.3}$$

or equivalently

$$\theta_k = \left\lfloor k! \left(x - \sum_{n=2}^{k-1} \frac{\theta_n}{n!} \right) \right\rfloor \tag{2.4}$$

We claim that

- (a) $0 \leq \theta_k < k$
- (b) $\left| x - \sum_{n=2}^k \frac{\theta_n}{n!} \right| < \frac{1}{k!}$

If $\theta_k \geq k$ then we can increase θ_{k-1} by 1 since

$$\frac{\theta_{k-1}}{(k-1)!} + \frac{\theta_k}{k!} = \frac{\theta_{k-1} + 1}{(k-1)!} + \frac{\theta_k - k}{k!}$$

which contradicts 2.2. This proves (a).

If $\left| x - \sum_{n=2}^k \frac{\theta_n}{n!} \right| \geq \frac{1}{k!}$ then we can increase θ_k to $\theta_k + 1$ and still have

$$\sum_{n=2}^{k-1} \frac{\theta_n}{n!} + \frac{\theta_k + 1}{k!} \leq x$$

contrary to 2.2 This proves (b). \square

Corollary 2.3: With the above definition of the θ_k , the sum $\sum_{n=2}^k \frac{\theta_n}{n!}$ converges to x as $k \rightarrow \infty$.

For brevity write

$$[\theta_2, \theta_3, \dots] = \sum_{n=2}^{\infty} \frac{\theta_n}{n!}$$

The expansion is not unique, but we can characterise the lack of uniqueness. First, observe:

Lemma 2.4: We have

$$\sum_{n=k}^{\infty} \frac{n}{(n+1)!} = \frac{1}{k!}$$

Proof: The sum is telescoping:

$$\begin{aligned} &= \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} + \frac{k+2}{(k+3)!} + \dots \\ &= \frac{k+1}{(k+1)!} - \frac{1}{(k+1)!} + \frac{k+2}{(k+2)!} - \frac{1}{(k+2)!} + \frac{k+3}{(k+3)!} - \frac{1}{(k+3)!} + \dots \\ &= \left[\frac{1}{k!} + \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \dots \right] - \left[\frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \frac{1}{(k+3)!} + \dots \right] \\ &= \frac{1}{k!} \end{aligned} \quad \square$$

Proposition 2.5: Suppose that for some $k \geq 2$ we have $\theta_{k-1} < k - 1$, but $\theta_\ell = \ell - 1$ for all $\ell \geq k$. Then

$$[\theta_2, \theta_3, \dots, \theta_{k-1}, \theta_k, \theta_{k+1}, \dots] = [\theta_2, \theta_3, \dots, \theta_{k-1} + 1, 0, 0, \dots]$$

Proof: With the stated conditions, Lemma 2.4 implies that

$$\frac{\theta_{k-1} + 1}{(k-1)!} = \frac{\theta_k}{k!} + \frac{\theta_{k+1}}{(k+1)!} + \dots \quad \square$$

It follows that every terminating factorimal has an alternative non-terminating expansion. We claim this is the only case where a factorimal expansion is not unique.

Theorem 2.6: Suppose that

$$\sum_{n=2}^{\infty} \frac{\theta_n}{n!} = \sum_{n=2}^{\infty} \frac{\phi}{n!} \quad (2.5)$$

where $\theta_n, \phi_n \in \mathbb{N}$ and $0 \leq \theta_n, \phi_n < n$. Then either $\theta_n = \phi_n$ for all $n \geq 2$, or there exists $k \geq 2$ such that

$$\begin{aligned} \theta_n &= \phi_n \text{ for all } n \text{ with } 2 \leq n \leq k, \\ \phi_k &= \theta_k + 1, \\ \theta_n &= n - 1 \text{ for all } n > k, \\ \phi_n &= 0 \text{ for all } n > k, \end{aligned}$$

or the same conditions hold with all θ_n and ϕ_n interchanged.

Proof: Suppose that (2.5) holds with the stated conditions. If $\theta_n \neq \phi_n$ for some n , let k be the smallest integer for which $\theta_k \neq \phi_k$. Interchanging all θ_n and ϕ_n if necessary, we may assume that $\theta_k < \phi_k$. Then

$$\sum_{n=k}^{\infty} \frac{\theta_n}{n!} = \sum_{n=k}^{\infty} \frac{\phi}{n!} \quad (2.6)$$

and $\theta_k \leq \phi_k - 1$. The left hand side of (2.6) is less than or equal to

$$\frac{\phi_k - 1}{k!} + \sum_{n=k+1}^{\infty} \frac{n-1}{n!} = \frac{\phi_k - 1}{k!} + \frac{1}{k!} = \frac{\phi_k}{k!}$$

with equality if and only if $\theta_n = n - 1$ for all $n > k$. Here we have used Lemma 2.4. The right hand side of (2.6) is greater than or equal to $\frac{\phi_k}{k!}$. Therefore $\theta_n = n - 1$ for all $n > k$ and $\phi_n = 0$ for all $n > k$. \square

Corollary 2.7: Every rational in $[0,1)$ has a unique terminating factorimal expansion.

The Irrationality Machine

Now we turn factorimals into dynamics. Suppose that A is an attractor of a dynamical system. Then the *basin of attraction* $\beta(A)$ of A is the set of all points $x(0)$ such that $x(t) \rightarrow A$ as $t \rightarrow +\infty$. Informally, this comprises all points whose trajectories approach indefinitely close to A in forward time.

The basins of attraction of the system's various attractors partition the phase space into disjoint regions, with the exception of points that lie on basin boundaries. It turns out that the geometry of the basins can be topologically wild. The long-term

behaviour of points on (and in practice near) the basin boundaries then becomes unpredictable: it is not feasible to decide which attractor the point's trajectory approaches. So we may be able to say that the long term behaviour can be one among several possibilities, while being unable to say which.

Systems with bizarre behaviour need not have complex equations. Consider, for example, the time-dependent discrete dynamical system

$$x_{t+1} = (t+1)x_t \pmod{1}$$

whose phase space is the circle $\mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$. We call this the *irrationality machine*, for reasons we now explain.

Geometrically, the dynamic wraps the circle round itself increasingly many times, so this system is unlikely to occur in any realistic physical model. But its mathematical structure has some interest.

The dynamic is a non-invertible map, so the system is a semi-dynamical system, defined only for $t \geq 0$. Distinct initial points can have orbits that become *identical* after some finite time, rather than merely converging. For example, $x_0 = 0$ and $x_0 = 1/2$ both lead to $x_1 = 0$. There is an explicit solution: clearly

$$x_t = (t!)x_0$$

Therefore all rationals tend to 0, indeed, reach 0 after finitely many iterations. If $x_0 = p/q$ with $p, q \in \mathbb{Z}$ and $q \neq 0$, then $x_q = (q!)p/q$, which is an integer, hence equal to 0 (mod 1). Conversely, if $x_t = 0$ then $(t!)x_0 \in \mathbb{Z}$ so x_0 is rational.

Since the effect of the factor $t + 1$ is not time-periodic, we do not expect to find periodic points. In fact, if $x_s = x_t$ for distinct s, t then $(s!)x_0 = (t!)x_0$ so $(s! - t!)x_0 = 0$ modulo 1, implying that x_0 is rational. Therefore 0 is the unique periodic point.

Some irrational numbers have orbits that converge to 0 without actually reaching it. An example is e , whose fractional part is $e - 2$. Consider the power series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

and take $x_0 = e$. Then $x_t = (t!)e$, which is of the form

$$\begin{aligned} (t!)e &= (t!)\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{t!}\right) + \left(\frac{1}{t+1} + \frac{1}{(t+1)(t+2)} + \cdots\right) \\ &= n + \left(\frac{1}{t+1} + \frac{1}{(t+1)(t+2)} + \cdots\right) \quad (n \in \mathbb{Z}) \\ &= \frac{1}{t+1} + \frac{1}{(t+1)(t+2)} + \cdots \pmod{1} \end{aligned}$$

The series has bounds

$$\frac{1}{t+1} \leq \frac{1}{t+1} + \frac{1}{(t+1)(t+2)} + \cdots \leq \frac{1}{t+1} + \frac{1}{(t+1)^2} + \cdots = \frac{1}{t}$$

Therefore $x_t \notin \mathbb{Z}$ for any t , but $x_t \rightarrow 0$ as $t \rightarrow \infty$.

As a corollary, we have proved that e is irrational. In fact, the calculation is a thinly disguised version of the standard proof of this property, see for example Hardy [2] Example XCI.6, page 343.

The number e is not alone in possessing this property. Another easy example is e^{-1} . The proof uses the identity

$$1/e = 1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \frac{4}{4!} - \frac{5}{5!} + \cdots = \frac{2}{3!} + \frac{4}{5!} + \cdots$$

The same method applies more widely.

Of course more is known: e^q is irrational for all nonzero rational q , see [1].

References

- [1] M. Aigner and G. Ziegler, *Proofs from THE BOOK* (4th ed.), Springer, Berlin 2009.
- [2] G.H. Hardy, *Pure Mathematics* (10th ed.), Cambridge University Press, Cambridge 1960.
- [3] http://en.wikipedia.org/wiki/Proof_that_e_is_irrational (Proof that e is irrational).

Trisecting the Angle

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The first ruler and compass constructions date back to the time of the ancient Greeks. They set the rules and many discoveries were made by Euclid and Archimedes. However, there are problems they couldn't solve. The two most famous of these are the trisection of an angle (i.e. dividing an arbitrary angle into three equal parts) and squaring the circle (constructing a square with area equal to the area of a given circle) using only a ruler and a compass. These problems remained unsolved for 2000 years until it was proved in the 19th century that the constructions are impossible [1].

In this article I will present a proof of the impossibility of trisecting an arbitrary angle. This was first proved in 1837 by Pierre Wantzel [1]. The proof is based on Cartesian coordinates – our “usual” coordinate system which was not known to the ancient Greeks.

Lots of definitions

To start the proof, the rules have to be specified formally. Let X be a finite set of points in \mathbb{R}^2 . Then the following two operations are allowed:

- (i) draw a line through two distinct points of X ;
- (ii) draw a circle passing through $a \in X$ with centre $b \in X$.

Note that we are not allowed to use any markings on the ruler. It is therefore sometimes also called a straightedge.

We shall say that a point P in \mathbb{R}^2 is *obtainable by ruler and compass in one step from the set X* if P is a point of intersection of:

- two distinct lines obtained using operation (i);
- two distinct circles obtained using operation (ii);
- one such line and one such circle.

A point P is *constructible* if there is a finite sequence of points $P_1, P_2, \dots, P_n = P$ such that P_1 is obtainable by ruler and compass in one step from $X_0 = \{(0,0), (1,0)\}$,

and such that P_{i+1} is obtainable by ruler and compass in one step from the set $X_i = \{(0,0), (1,0), P_1, \dots, P_i\}$ for $i = 1, \dots, n-1$.

A number x is called a *constructible number* if $(x,0)$ is a constructible point.

Which points can't be constructed?

Let F be a subfield of \mathbb{R} (i.e. a set of real numbers that is closed with respect to addition, subtraction, multiplication, and division just like \mathbb{R} or \mathbb{Q} . For example, $a, b \in F \Rightarrow a + b \in F$) and define $F(\sqrt{x}) := \{a + b\sqrt{x} : a, b \in F\}$. This is called a *quadratic extension* of F and is itself a subfield of \mathbb{R} .

Let S be the set of all real numbers that can be obtained from the integers using only the four basic arithmetic operations $+$, $-$, \times , $/$ and square roots. So for example, $\sqrt{\frac{1}{2} - \frac{3}{\sqrt{4+\sqrt{7}}}}$ is in S .

In order to prove that all constructible numbers are in S , we need a few lemmas (taken from [2]).

Lemma 1. If a line passes through two points each having coordinates in field F , then the line has an equation with coefficients in F . If both the centre of a circle and a point on the circle have coordinates in field F , then the circle has an equation with coefficients in F .

Proof. We can derive lemma 1 just by writing out the equation of the line and circle: The equation of a line through (x_1, y_1) and (x_2, y_2) is $(y_2 - y_1)X - (x_2 - x_1)Y + (x_2y_1 - x_1y_2) = 0$ and the equation of a circle with centre (p, q) that passes through (s, t) is $(X - p)^2 + (Y - q)^2 = (s - p)^2 + (t - q)^2$ which is equivalent to $X^2 + Y^2 + (-2p)X + (-2q)Y + (s(2p - s) + t(2q - t)) = 0$. Now use the fact that fields are closed under the four basic arithmetic operations. \square

Lemma 2. If each of two intersecting lines has an equation with coefficients in field F , then the point of intersection has coordinates in F . \square

Lemma 3. If a line and a circle intersect and each has an equation with coefficients in field F , then the points of intersection have coordinates in F or in a quadratic extension of F .

Proof. The line with equation $aX + bY + c = 0$ and the circle with equation $X^2 + Y^2 + fX + gY + h = 0$ intersect at the points (x_0, y_0) where

$$d = (fb - ag)^2 + 4c(af + gb - c) - 4h(a^2 + b^2)$$

$$x_0 = \frac{abg - 2ac - b^2f \pm b\sqrt{d}}{2(a^2 + b^2)}, \quad y_0 = \frac{abf - 2bc - a^2g \mp a\sqrt{d}}{2(a^2 + b^2)}.$$

We suppose a, b, c, f, g, h are in F . In order for the line and circle to intersect, d must be nonnegative. \square

Lemma 4. If each of two intersecting circles has an equation with coefficients in field F , then the points of intersection have coordinates in F or in a quadratic extension of F . \square

I didn't write down the proofs for lemmas 2 and 4 because they are similar to the proof of lemma 3, and they are similarly uninspiring.

A point in \mathbb{R}^2 is constructible if and only if both coordinates are constructible numbers, so starting with $F = \mathbb{Q}$ it follows from these lemmas that all constructible numbers are in the set S defined above. This is useful, because if we find a number that is not in S , then we know that it can't be constructible. In fact, all numbers in S are constructible, but that isn't important here.

The proof

Now we get to the actual problem of trisecting an angle. We want to prove that there is no general method using only ruler and compass with which every angle can be trisected. In order to show this it suffices to find a particular angle that can't be trisected. First of all, we need another short lemma.

Lemma 5. The angle x is constructible if and only if $\cos(x)$ is a constructible number.

Proof. See Figure 1 below. \square

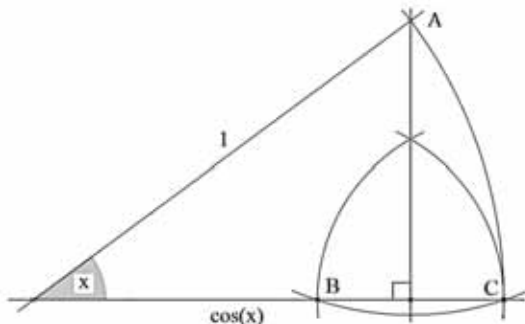


Figure 1: The circle through B and C has centre A.

Now all that remains to be done is to show that $\cos(\frac{\pi}{9}) \notin S$. If we succeed in proving this, then we have found an angle $\theta = \frac{\pi}{3}$ that can be constructed (see figure 2), but $\frac{\theta}{3}$ is an angle that can't be constructed by Lemma 5. This proves the hypothesis.

Recall the trigonometric formulae

$$\begin{aligned}\sin(a + b) &= \sin(a) \cos(b) + \cos(a) \sin(b), \\ \cos(a + b) &= \cos(a) \cos(b) - \sin(a) \sin(b).\end{aligned}$$

Combining these, we get

$$\cos(3a) = \cos(2a) \cos(a) - \sin(2a) \sin(a) = 4 \cos^3(a) - 3 \cos(a).$$

So substituting $a = \frac{\pi}{9}$ and $x = \cos(a)$ this becomes $\frac{1}{2} = 4x^3 - 3x$ which is equivalent to $8x^3 - 6x - 1 = 0$.

It is intuitively clear that the zeros of this cubic polynomial are not in S because they should involve cube roots. In fact, $x \in S \Rightarrow x$ satisfies a minimal polynomial with rational coefficients whose degree is a power of 2. However, the proof of this requires knowledge from Galois theory, so I will not give it here (see [2] and [3]). In any case, it follows that $\cos(\frac{\pi}{9}) \notin S$, which completes the proof!

So I will never be able to trisect angles?

In the course of history different sets of rules have been tried out, and some of them do allow angles to be trisected. For example, if we work with a marked ruler instead of the unmarked straightedge, then it is possible. Archimedes gave such a construction which uses a ruler that has only two marks on it.

Likewise, if we allow ourselves to construct with straightedge and compass shapes that can be moved around on the plane, then there are constructed “tools” that make angle trisection possible. See [4] for more information. Finally, it is possible to trisect acute angles using origami (i.e. folding paper). See [5] for instructions.

References and Further Reading

- [1] <http://mathforum.org/dr.math/faq/faq.impossible.construct.html/>.
- [2] Martin, G. E., *Geometric Constructions*, Springer Verlag, Berlin, 1998.
- [3] Ellis, G., *Rings and Fields*, Oxford University Press, Oxford, 1992.
- [4] <http://www.jimloy.com/geometry/trisect.htm/>.
- [5] <http://www.math.lsu.edu/~verrill/origami/trisect/>.
- [6] Kazarinoff, N. D., *Ruler and the Round: or Angle Trisection and Circle Division*, Prindle, Weber & Schmidt, Inc., Boston, 1970.

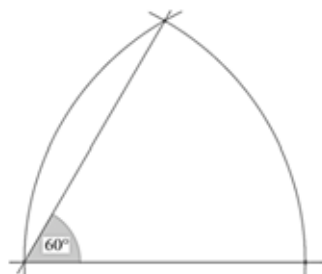


Figure 2: Equilateral triangle

Dissecting Parallelotopes

Anja Komatar, Queens' College

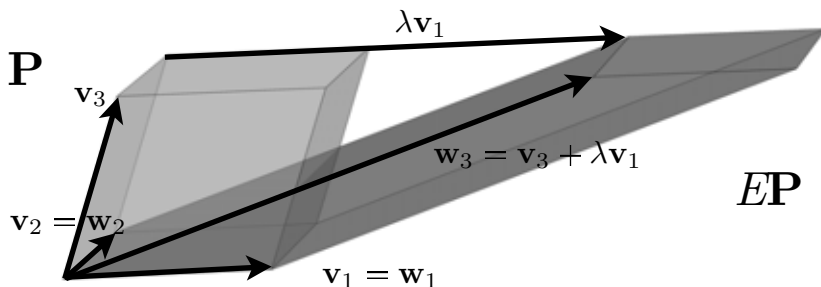
Two bodies are said to be scissor-congruent if one can be cut into finitely many pieces, that can be rearranged to form the other. By the Bolyai-Gerwien theorem any two polygons of the same area are scissor-congruent. Hilbert's third problem is related to its 3D analogue — it asks whether or not two polyhedra of same volume are scissor congruent. By considering Dehn's invariant one can show that a cube and a tetrahedron are not, so the question about scissor congruency of polytopes becomes interesting.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be n -dimensional vectors and $\mathbf{V} = (\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n)$ an $n \times n$ matrix. If $\det \mathbf{V} \neq 0$, then $\mathbf{P} = \{\sum_{i=1}^n \mu_i \mathbf{v}_i, \mu_i \in [0,1]\}$ is an n -parallelotope described by vectors \mathbf{v}_i . It's volume is $|\det \mathbf{V}|$. This article shows that any parallelotopes of the same volume are scissor congruent.

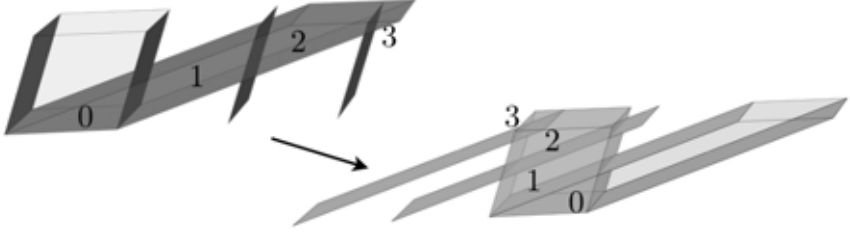
This can be shown by using elementary geometry. To start with, define n by n matrices $E(\lambda, r, s)_{ij} = \delta_{ij} + \lambda \delta_{ir} \delta_{js}$ and $F(\nu, r, s)_{ij} = \delta_{ij} + (\nu - 1) \delta_{ir} \delta_{jr} + (\frac{1}{\nu} - 1) \delta_{is} \delta_{js}$ with $r \neq s$ and $\nu > 0$. Observe that $|\det E| = |\det F| = 1$.

Define also an $(n - 1)$ -dimensional hyperplane by

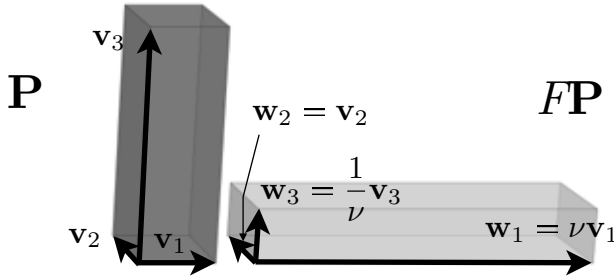
$$H(\lambda, r, s, k) = \{\sum_{i \neq r, s} \mu_i \mathbf{v}_i + \mu_s (\mathbf{v}_s + \lambda \mathbf{v}_r) + k \mathbf{v}_r, \mu_i \in \mathbb{R}\}.$$



Lemma 1: Parallelotopes \mathbf{P} and \mathbf{EP} , described by vectors in \mathbf{V} and $E(\lambda, r, s)\mathbf{V}$ respectively, are scissor congruent.

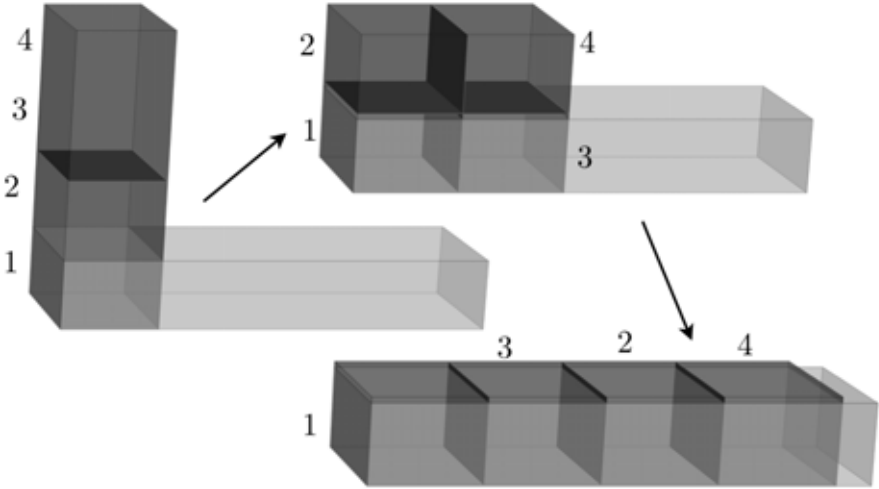


Proof: Given λ , r and s , consider all k such that $H(\lambda, r, s, k) \cup \mathbf{V} \neq \emptyset$. Cut \mathbf{V} along all such $H(\lambda, r, s, k)$ to get pieces of \mathbf{V} , k^{th} piece lying between $H(\lambda, r, s, k)$ and $H(\lambda, r, s, k + 1)$. Clearly there are $\lceil \lambda \rceil + 1$ such k 's, as \mathbf{P} meets $H(\lambda, r, s, 0)$ in $\mathbf{0}$ and $H(\lambda, r, s, -\lambda)$ in point \mathbf{v} , but none of $H(\lambda, r, s, k)$ with $|k| > \lceil \lambda \rceil + 1$. Now translate k^{th} piece by vector $-k\mathbf{v}_r$. Then it lies between $H(\lambda, r, s, 0)$ and $H(\lambda, r, s, 1)$. The pieces form a parallelepiped $\mathbf{P}' = E\mathbf{P}$. Indeed, for all $j \neq r$, both \mathbf{V} and $E(\lambda, r, s)\mathbf{V}$ lie between $H(0, j, r, 0)$ and $H(0, j, r, 0)$, whereas \mathbf{V} lies between $H(0, r, s, 0)$ and $H(0, r, s, 1)$ and $E(\lambda, r, s)\mathbf{V}$ lies between $H(\lambda, r, s, 0)$ and $H(\lambda, r, s, 1)$. Thus for any $x \in \mathbf{V}$, x is in the k^{th} piece and gets translated to $x' \in E(\lambda, r, s)\mathbf{V}$, so $\mathbf{P}' \subset E\mathbf{P}$. But by cutting \mathbf{P} in finitely many pieces and rearranging them to form \mathbf{P}' we have not changed it's volume, which is equal to the volume of $E\mathbf{P}$. Thus $\mathbf{P}' = E\mathbf{P}$, so \mathbf{P} and $E\mathbf{P}$ are scissor congruent. \square



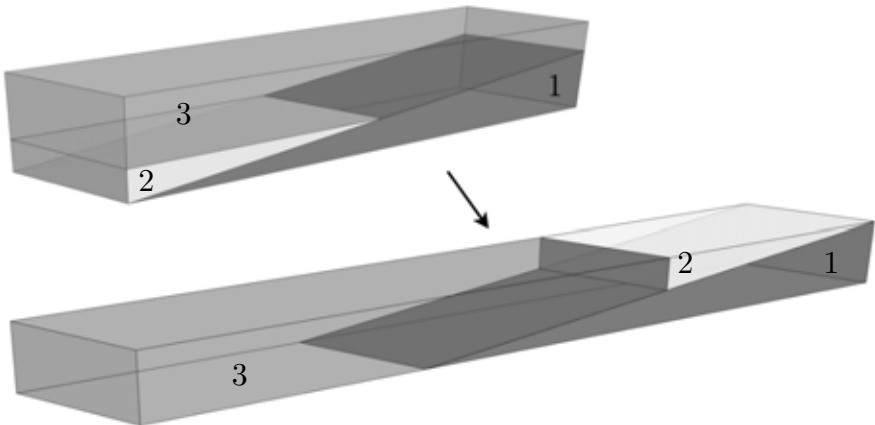
Lemma 2: Parallelotopes \mathbf{P} and $F\mathbf{P}$, described by vectors in \mathbf{V} and $F(\nu, r, s)\mathbf{V}$ respectively, are scissor congruent.

Proof: The lemma is obvious for $\mathbf{v}_i \parallel F\mathbf{v}_i \ \forall i$. Otherwise, there exists $z \in \mathbb{Z}$ such that $2^{z-1} < \nu \leq 2^z$. Relabel $\mathbf{P} = \mathbf{P}^{(0)}$. If $z > 0$, form $\mathbf{P}^{(k)}$ by cutting $\mathbf{P}^{(k-1)}$ along $H(0, s, r, 0.5)$ and translating the piece of \mathbf{P} , that does not contain $\mathbf{0}$ by $\mathbf{v}_r^{(k-1)} - 0.5\mathbf{v}_s^{(k-1)}$, until getting $\mathbf{P}^{(z)}$. If $z < 0$ proceed similarly, but exchanging r and s , to get $\mathbf{P}^{(-z)}$. In each step we get $\mathbf{v}_r^{(k)} = 2^{\text{sign}(z)}\mathbf{v}_r^{(k-1)}$ and $\mathbf{v}_s^{(k)} = 2^{-\text{sign}(z)}\mathbf{v}_s^{(k-1)}$.



So in each case we get $\mathbf{P}^{(|z|)} = F(2^z, r, s)\mathbf{P}$, so \mathbf{P} and $F(2^z, r, s)\mathbf{P}$ are scissor congruent.

Now we want to show that $\mathbf{P}^{(|z|)}$ and $F(\nu, r, s)\mathbf{P} = F(\nu', r, s)\mathbf{P}^{(|z|)}$, where $0.5 < \nu' \leq 1$, are scissor congruent. If $\nu' = 1$ that is obvious. Else cut $\mathbf{P}^{(|z|)}$ along $H(-\nu', r, s, \nu')$ to get piece 1, containing $\mathbf{0}$. Cut the other piece along $H(0, r, s, 1 - \nu')$ to get piece 2 on the same side of $H(0, r, s, 1 - \nu')$ as $\mathbf{0}$ and piece 3 on the other. Then translate piece 2 by $(2 - \frac{1}{\nu'}) (\nu' \mathbf{v}_r - \mathbf{v}_s)$ and piece 3 by $(\frac{1}{\nu'} - 1) (\mathbf{v}_s - \nu' \mathbf{v}_r)$. We get $F(\nu', r, s) \mathbf{P}^{(|z|)}$, which is clearly scissor congruent with $\mathbf{P}^{(|z|)}$. Thus also \mathbf{P} and $F\mathbf{P}$ are scissor congruent as required. \square



Theorem: Any two n -parallelotopes of same volume are scissor congruent.

Proof: That is to say, if \mathbf{P} and \mathbf{R} are n -parallelotopes described by vectors \mathbf{v}_i and \mathbf{w}_i respectively, and $|\det \mathbf{V}| = |\det \mathbf{W}|$, then we can cut \mathbf{P} in finitely many pieces and rearrange them to form \mathbf{R} . This will be shown in two steps. At first \mathbf{P} will be cut and its pieces rearranged to form \mathbf{P}' , described by vectors \mathbf{v}'_i , such that $\mathbf{v}'_i \parallel \mathbf{w}_i$ for all i . Then \mathbf{P}' will be cut to pieces that can form \mathbf{R} .

Since \mathbf{v}_i and \mathbf{w}_i are linearly independent (else $|\det \mathbf{V}| = 0$ or $|\det \mathbf{W}| = 0$), they form two sets of basis of \mathbb{R}^n . Then λ_i in $\mathbf{w}_1 = \sum_{i=1}^n \lambda_i \mathbf{v}_i$ are uniquely defined and not all zero. Switch the labels of \mathbf{v}_j and \mathbf{v}_1 for the smallest j such that $\lambda_j \neq 0$, so that now $\mathbf{w}_1 = \lambda_1 \mathbf{v}_1 + \sum_{i=2}^n \lambda_i \mathbf{v}_i$, $\lambda_1 \neq 0$. Apply Lemma 1 with $E(\frac{\lambda_j}{\lambda_1}, j, 1)$ for all $j > 1$. Applying Lemma 1 does not affect any \mathbf{v}_i with $i \neq 1$, but shows how to dissect \mathbf{P} to a parallelotope, described by vector

$$\mathbf{v}'_1 = \mathbf{v}_1 + \sum_{j=2}^n \frac{\lambda_j}{\lambda_1} \mathbf{v}_j = \frac{1}{\lambda_1} (\mathbf{w}_1 - \sum_{j=2}^n \lambda_j \mathbf{v}_j) + \sum_{j=2}^n \frac{\lambda_j}{\lambda_1} \mathbf{v}_j = \frac{1}{\lambda_1} \mathbf{w}_1$$

so $\mathbf{P}^{(1)}$ is described by vectors \mathbf{v}'_1 and \mathbf{v}_i for $i > 1$, with $\mathbf{v}'_1 \parallel \mathbf{w}_1$. Proceed similarly to construct \mathbf{v}'_j for $1 < j \leq n$. Note that having constructed \mathbf{v}'_i for $i < j$, \mathbf{v}'_i together with \mathbf{v}_i for $i \geq j$ form a basis of \mathbb{R}^n , and since \mathbf{w}_i are linearly independent, $\mathbf{w}_j = \sum_{i=1}^{j-1} \lambda_i \mathbf{v}'_i + \sum_{i=j}^n \lambda_i \mathbf{v}_i$ with at least one of λ_i for $i \geq j$ nonzero, so we can switch labels of that \mathbf{v}_i and \mathbf{v}_j . Thus we have cut \mathbf{P} and rearranged its pieces to construct $\mathbf{P}^{(n)} = \mathbf{P}'$, with vectors describing it parallel to those of \mathbf{R} .

Now we have $\mathbf{w}_i = \nu'_i \mathbf{v}'_i$ for all i . Translate \mathbf{P}' by $-\sum_k \mathbf{v}'_k$ for all k with $\nu'_k < 0$ to get $\mathbf{P}^{(1)}$, described by vectors $\mathbf{v}^{(0)} = \text{sgn}(\nu'_i) \mathbf{v}'_i$ with $\mathbf{w}_i = \nu_i \mathbf{v}_i^{(0)}$ and $\nu_i > 0$. Having constructed $\mathbf{P}^{(k-1)}$, construct $\mathbf{P}^{(k)}$ by applying Lemma 2 with $F(\nu_k, k, k+1)$ for $1 \leq k < n$. In each step $\mathbf{P}^{(k)}$ is described by vectors $\mathbf{v}^{(k)}_k = \nu_k \mathbf{v}^{(k-1)}_k = \mathbf{w}_k$, $\mathbf{v}^{(k)}_{k+1} = \frac{1}{\nu_k} \mathbf{v}^{(k-1)}_{k+1}$ and $\mathbf{v}^{(k)}_i = \mathbf{v}^{(k-1)}_i$ for $i \neq k, k+1$. Since volume is preserved in each step, we must have $\mathbf{v}^{(n-1)}_n = \mathbf{w}_n$ and $\mathbf{P}^{(n-1)} = \mathbf{R}$. \square

References

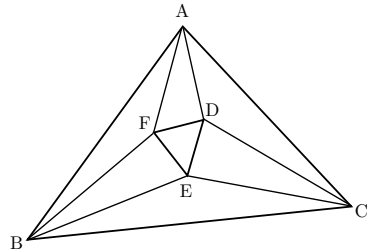
- [1] http://en.wikipedia.org/wiki/Hilbert's_third_problem.
- [2] S.J. Cowley: Vectors and Matrices lecture notes (Michaelmas 2009).

Morley's Theorem

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The past three hundred years have seen the birth of different non-Euclidean geometries, such as elliptic and hyperbolic geometry, and the introduction of mathematical rigour into Euclidean geometry, such as the proof of Euclid's Fifth Postulate, the "Parallel Postulate", from his first four postulates.

However, in 1899, Frank Morley (8th Wrangler, BA in Mathematics, 1884, King's College, Cambridge), then professor of mathematics at Haverford College, US, discovered a surprising result in plane geometry, the so-called Morley's Miracle. The fact that it has never been discovered before is mysterious. It states that



The three points of intersection of the adjacent trisectors of the angles of any triangle form an equilateral triangle.

Morley's original proof revolved around algebraic curves. Subsequently, many attempts were made to find elementary proofs which would match the level of knowledge sufficient to understand the theorem itself. A number of such proofs have been found, most of which involve trigonometric identities or backward methods. Interestingly, two famous Cambridge mathematicians, Béla Bollobás and John Conway, have their own proofs to the theorem. Alan Connes, a French mathematician (Fields Medallist, 1982), also holds an innovative proof of it. If you have not come across this theorem before, probably you can give it a try yourself!

It is interesting to note that Morley's Theorem does not hold in spherical and hyperbolic geometry and that the angle trisectors of exterior and interior angles give a total of five equilateral triangles.

Although the theorem is more than one hundred years old, new proofs are still coming out recently. Here I will present two proofs, one using trigonometric identity and the other using backward method.

First Proof

Note the trigonometric identity $\sin 3\theta = 4 \sin \theta \sin (60^\circ + \theta) \sin (120^\circ + \theta)$. Points D , E and F are constructed on BC as shown. Let $\angle A = 3a$, $\angle B = 3b$ and $\angle C = 3c$. Clearly $\angle AYC = 120^\circ + b$. Apply the Sine Rule on $\triangle AYC$,

$$\frac{AC}{\sin(120^\circ + b)} = \frac{AY}{\sin c}.$$

The height of $\triangle ABC$ with base BC is

$$\begin{aligned} h &= AB \sin 3b = \frac{4AB \times AC \times DX}{XE \times AY} \sin b \sin c \\ &= AC \sin 3c = \frac{4AC \times AB \times DX}{XF \times AZ} \sin c \sin b. \end{aligned}$$

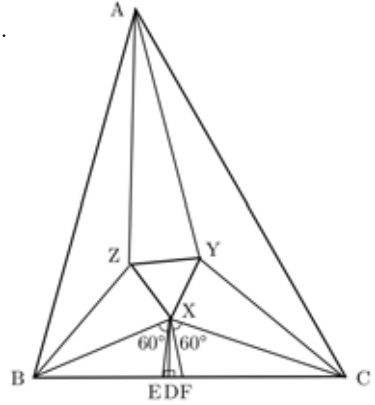
Since the numerators are equal,

$$XE \times AY = XF \times AZ.$$

But $\angle ZAY = \angle EXF = a$, so $\angle ZAY$ and $\angle EXF$ are similar.

$$\begin{aligned} \angle AZY &= 60^\circ + b, \\ \angle AYZ &= 60^\circ + c. \end{aligned}$$

All other angles in the diagram can be determined similarly, hence proving the theorem. \square



Conway's proof

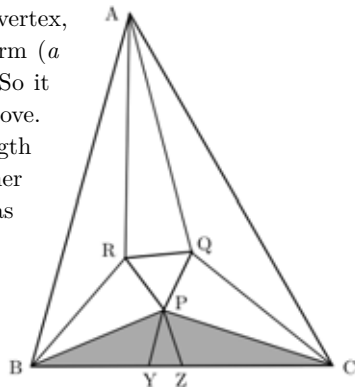
Let the triangle have angles $3a$, $3b$, $3c$ and let x' mean $x + 60^\circ$. Then the triangles with angles $(0', 0', 0')$, (a, b', c') , (a', b, c') , (a', b', c) , (a'', b, c) , (a, b'', c) and (a, b, c'') exist abstractly, as the sum of angles is 180° in each case. Build them in such a way:

- $(0', 0', 0')$: This is equilateral. Make it have edge of length 1.
- (a, b', c') : Make the edge joining the angles b' and c' have length 1. Similarly for (a', b, c') and (a', b', c) .
- (a'', b, c) : Shown as shaded triangle below. Let the angles at B , P , C be b , a'' , c . Draw lines from P cutting BC at Y and Z such that $PZ = 1$ and $\angle PYC = \angle PZB = a'$. Similarly for (a, b'', c) and (a, b, c'') .

Now we can fit all these 7 triangles together to form the figure on the following page. Corresponding points to Y , Z for the other two triangles are omitted. In which case $\angle RBP = b$, $\angle RPB = c'$, $\angle PRB = a'$.

It can be easily checked that at each internal vertex, the angles add up to 360° . Also, $\angle BPZ$ is of the form (a', b, c') and the edge joining b' and c' has length 1. So it is congruent to the triangle (a', b, c') drawn above. Hence, the edge joining b and c' have the same length for the two triangles. Hence, adjacent edges fit either because of this argument or both have length 1 as declared.

This figure (ignoring PY and PZ) is similar to original triangle given. Hence, the middle sub-triangle must also be equilateral. \square



References

- [1] Morley's Trisector Theorem on Wikipedia:
http://en.wikipedia.org/wiki/Morley's_trisector_theorem.
- [2] Morley's Theorem on Cut-the-Knot:
<http://www.cut-the-knot.org/triangle/Morley/>.

A brief Note on Doubles Tournaments

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The purpose of this note is to describe a short solution of the Doubles Tournament problem and the Spouse-Avoiding Doubles Tournament problem for $4k$ and $2k$ players respectively, when k is a power of 2. Similar tournaments are known for almost all values of k — see [1] and [2].

The Doubles Tournament Problem: Given $4k$ players, to arrange a tournament of $(4k - 1)$ rounds, each of k double matches, so that any two players play with each other exactly once and against each other exactly twice.

Rule 1: Let F be a finite field of order $4k$. We index the players by the members of F and the rounds by the set F' of non-zero elements of F . Pick $\xi \in F$, with $\xi \neq 0$ or 1. In round ψ , player θ will play with player $\theta + \psi\xi$ against players $\theta + \psi$ and $\theta + \psi + \psi\xi$.

Proof: To see that Rule 1 is coherent, note that the sets $\{\theta, \theta + \psi\xi\}$ are the cosets of the additive subgroup $\{0, \psi\xi\}$, itself a subgroup of $\{0, \psi, \psi\xi, \psi + \psi\xi\}$; of which larger subgroup the matches $\{\theta, \theta + \psi, \theta + \psi\xi, \theta + \psi + \psi\xi\}$ are the cosets.

The Spouse-Avoiding Doubles Tournament Problem: Given $2k$ married couples ($k > 1$), to arrange a tournament of $(2k - 1)$ rounds, each of k mixed doubles, so that no couple ever play in the same game, but such that each person plays *against* each other person, spouse excepted, exactly once and plays *with* each person of the other sex, spouse excepted, exactly once.

Rule 2: Let G be a finite field of $2k$ elements; index the couples by the members of G and the rounds by the set G' of non-zero elements of G . Pick $\xi \in G \setminus \{0, 1\}$ as before, and call the members of couple θ Man θ and Woman θ . In round ψ , Man θ and Woman $\theta + \psi\xi$ play Man $\theta + \psi$ and Woman $\theta + \psi + \psi\xi$.

The coherency and adequacy of this rule may be verified as above.

- [1] P. Healey, *Construction of Balanced Double Schedules*,
Journal of Combinatorial Theory Series A 29, 280-286, 1980.
- [2] W. D. Wallis, *Spouse-avoiding Mixed Double Tournaments*,
Annals of the New York Academy of Sciences, 549-554, 1979.

Music, Groups and Topology

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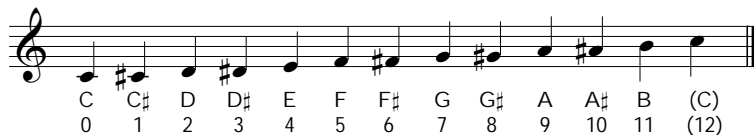
It is well known that many aspects of music can be explained using mathematics: music is created and propagates using sound waves with trigonometric functions, intervals are defined by ratios of frequencies, the idea of rhythm is based on multiples of certain time intervals and music is stored digitally on CDs.

In this article I want to examine the relationship between mathematics and music from the opposite point of view: We will see that there is music “hidden” in pure mathematics and that you can *hear* concepts such as group actions – in the same way as it is possible to *see* the symmetries of polygons in art.

Notes, Chords and Transformations

Pythagoras was the first to discover that two notes with a simple frequency ratio (such as $2/3$) sound *consonant*, while those with a more complicated frequency ratio (such as $17/24$) sound *dissonant*. Two notes with a frequency ratio of $1/2$ sound so similar that we can use them to divide the continuous scale of pitches into equivalence classes, which are called octaves.

In equal tempered tuning, each octave is divided into 12 equally spaced notes, with a ratio of $2^{1/12}$ between two consecutive notes. This is of course far from being a simple fraction, but a very close approximation.



In the following article it will be useful to replace these 12 notes by the group of integers mod 12, written as $\mathbb{Z}/12$ (since we don't distinguish between, for example, $C\sharp$ and $D\flat$). The style of music in which the composer explores the full chromatic scale (rather than just one major or minor key with 7 non-equally spaced notes) is called 12-tone music. It was developed during the time from R. WAGNER (1813–1883) to A. SCHÖNBERG (1874–1951).

We can also combine several notes to form chords. In particular we can consider the set S of all **consonant triads** which consists of

- *major triads* of the form $\{x, x + 3, x + 7\}$;
- *minor triads* of the form $\{x, x + 4, x + 7\}$.



Since $\mathbb{Z}/12$ is cyclic, we don't have to specify order of the three notes. However if the triad is of the form above (in root position) we write $\langle a, b, c \rangle$ and call notes *root*, *third* and *fifth* respectively. It is easy to see that S has size 24.

The group of symmetries of $\mathbb{Z}/12$ is the *dihedral group* of order 24, D_{24} . In music it is often called the ***T/I group*** because it consists of

- **Transpositions:** $T_n : x \mapsto x + n \pmod{12} \quad n \in \{0, \dots, 12\}$
- **Inversions:** $I_n : x \mapsto -x + n \pmod{12}$

Transposition and inversion can be found in many places in music, most notably the *Art of Fugue* by J. S. BACH. Inverting always swaps major and minor keys while transposing only changes the pitch. The *T/I group* acts both on the set of notes $\mathbb{Z}/12$ and the set of triads S . Clearly both those actions are transitive.

Geometrically we can identify the set of notes by a **regular 12-gon** and the triads by certain triangles connecting three of its vertices. Now the *T/I group* acts on the set of vertices and triangles by rotation and reflection. In figure 1 you can see the C-major triangle and its image under I_0 and T_2 .

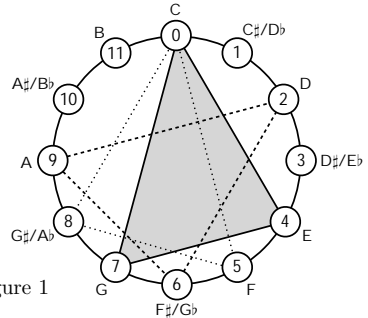


Figure 1

The *PLR* Group, the Tone Net and the Harmonic Torus

Although the *T/I group* is useful for analysing baroque and classical music, it doesn't give much insight into 19th and 20th century chromatic music. Therefore we also define the ***PLR-group*** or *neo-Riemannian group*, named after the music theorist HUGO RIEMANN (1894–1919). The *PLR-group* is generated by three functions P , L and R which we can define both musically and mathematically:

- **Parallel:** The operation P maps a major triad to its parallel minor and vice versa. Using the inversion operation defined above, $P\langle x, y, z \rangle = I_{x+z}\langle x, y, z \rangle$.
- **Leading tone exchange:** The operation L lowers the root note of a major triad by a semitone and raises the fifth of a minor triad by a semitone. Thus $L\langle x, y, z \rangle = I_{y+z}\langle x, y, z \rangle$.

- **Relative:** The operation R maps a major triad to its relative minor and vice versa, i.e. $R\langle x, y, z \rangle = I_{x+y}\langle x, y, z \rangle$.

All three operations take major chords to minor chords (and vice versa). They are musically interesting because they change precisely one of the three notes in a triad by a semitone. Therefore they enable *parsimonious voice leading*: the law of minimal motion of the moving voice. The following examples might help to visualise the actions of P , L and R :



$$\begin{array}{lll} P(0,4,7) = (0,3,7) & L(0,4,7) = (4,7,11) & R(0,4,7) = (9,0,4) \\ P(0,3,7) = (0,4,7) & L(0,3,7) = (8,0,3) & R(0,3,7) = (7,3,10) \end{array}$$

By applying the definitions in terms of inversions above it is not hard to check that the PLR -group is also dihedral of order 24 and in fact is generated by the transformations L and R only. As with the T/I group, the PLR -group acts simply transitively on the set of consonant triads S .

We can also find a geometrical representation of the PLR -group: the **tone net**. (figure 2). Each vertex represents a note and each triangle represents a major (\triangle) or a minor (∇) triad. P , L and R each move to one of the three adjacent triangles.

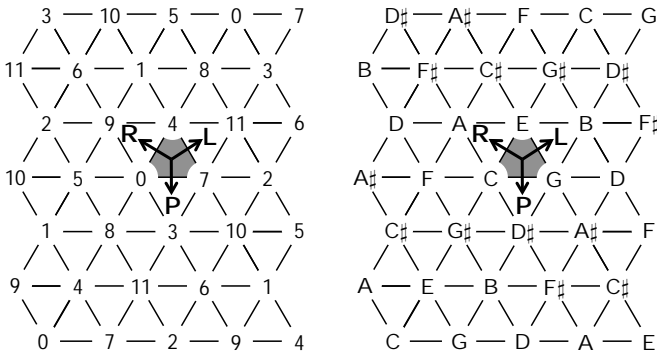


Figure 2

The note-net also shows other harmonic relationships between chords, such as dominant ($R \circ L$) and subdominant ($L \circ R$). Furthermore the keys on any horizontal axis sweep out the cycle of fifths.

Clearly the tone net can be extended in all directions. However it is more useful to connect opposite boundaries to form a **harmonic torus** (see figure 3 below).

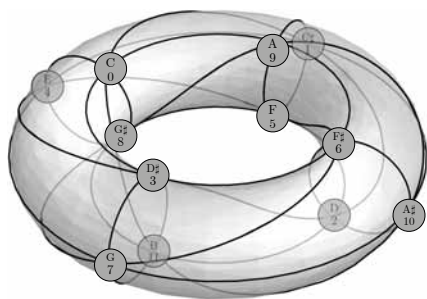


Figure 3

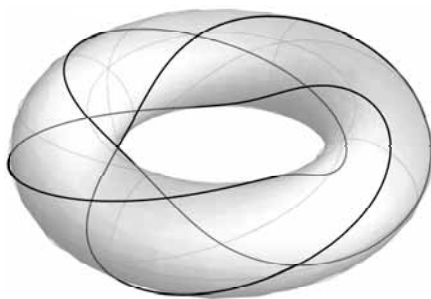


Figure 4

All harmonic progressions trace out a path on this torus. The most famous example probably is the following amazing extract from the second movement of L. VAN BEETHOVEN's 9th symphony which follows the path is shown in figure 4:

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C major —(R)—> A minor —(L)—> F major —(R)—> D minor —(L)—> B \flat major —(R)—> G minor —(L)—> E \flat major

—(R)—> C minor —(L)—> A \flat major —(R)—> F minor —(L)—> D \flat major —(R)—> B \flat minor —(L)—> G \flat major —(R)—> E \flat minor

—(L)—> B major —(R)—> A \flat minor —(L)—> E major —(R)—> C \sharp minor —(L)—> A major

[5 bars missing]

[5 bars missing]

Nearly all major and minor chords appear in this sequence. Mathematics gives us completely new (and very beautiful) ways of hearing the extract – which Beethoven wrote around 1822, 80 years before POINCARÉ initiated the subject of topology.

There are other similar examples, for example Bach's *Crab Canon* (from *The musical offering*) looks like a Möbius Strip.

The Duality of the T/I group and the PLR group

On the previous pages I have defined the T/I group and the PLR group. For each group there is a geometrical representations and there are examples of how they are used in music. They are both subgroups of $\text{Sym}(S)$, where S is the set of all consonant triads, which are isomorphic to D_{24} . However there is a very unexpected, much deeper relationship between these two groups, which is summarised in the following theorem (quoted from [3]):

Theorem: The T/I group and the PLR group are dual. That is, each acts simply transitively on the set S of consonant triads, and each is the centraliser of the other in the symmetric group $\text{Sym}(S)$.

Proof: The theorem, although surprising, is in fact not so unexpected when thinking about Cayley's Theorem. It states that any group G is isomorphic to a subgroup of $\text{Sym}(G)$. We can prove Cayley's theorem by considering the action of G on itself, either by left or by right multiplication. For $g, h \in G$ we define

$$\begin{array}{lll} p_g : G \rightarrow G & \text{with} & p_g(x) = gx & p_g \text{ is a permutation of } G; \\ q_h : G \rightarrow G & \text{with} & q_h(x) = xh & q_h \text{ is a permutation of } G. \end{array}$$

By Cayley's theorem, both $P = \{p_g : g \in G\}$ and $Q = \{q_h : h \in G\}$ under composition of functions are isomorphic to G . Also p_g and q_h commute for any $g, h \in G$ since

$$p_g(q_h(x)) = p_g(xh) = gxh = q_h(gx) = q_h(p_g(x)).$$

By considering the individual elements and how they act, it is not hard to show that we can identify the T/I group by P and the PLR group by Q . Since the elements in P and Q commute, we have $P \subseteq C_Q$ and $Q \subseteq C_P$, where C_X is the centraliser of X in $\text{Sym}(G)$. Therefore in fact $P = C_Q$ and $Q = C_P$, i.e. the T/I group and the PLR group are dual. \square

The fact that elements of the T/I and the PLR group commute can be used to analyse music even further. One element from each group, for example T_7 and R can be used to produce a commutative graph as shown below. The most famous piece related to this graph is the *Canon in D* by J. PACHELBEL:

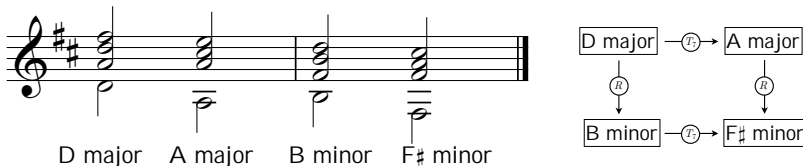


Figure 5

More Harmonies, more Symmetries...

Of course one could extend S to include dominant seventh, diminished, augmented and other more exotic chords. It is more difficult, but certainly not impossible to construct groups similar to T/I and PLR in these cases (the definitions of P , L and R in terms of inversions could be the same, but the musical interpretation is different). For 4-note chords, we would then get a 3-dimensional and tone net and a 4-dimensional torus. We can consider tuning systems where notes are not equally spaced.

Instead of looking at chromatic notes as points on a circle, we could consider the projective line with 11 elements together with a point at infinity. The projective linear group $\text{PGL}(2, \mathbb{Z}/11)$ acts on this line in the natural way and the action is sharply triply transitive. This means that $\text{PGL}(2, \mathbb{Z}/11)$, which has size $12 \times 11 \times 10 = 1320$, takes *any* 3-note chord to *any* other such chord. We can consider even more complicated groups which are quadruply or quintuply transitive: the **Mathieu Groups**:

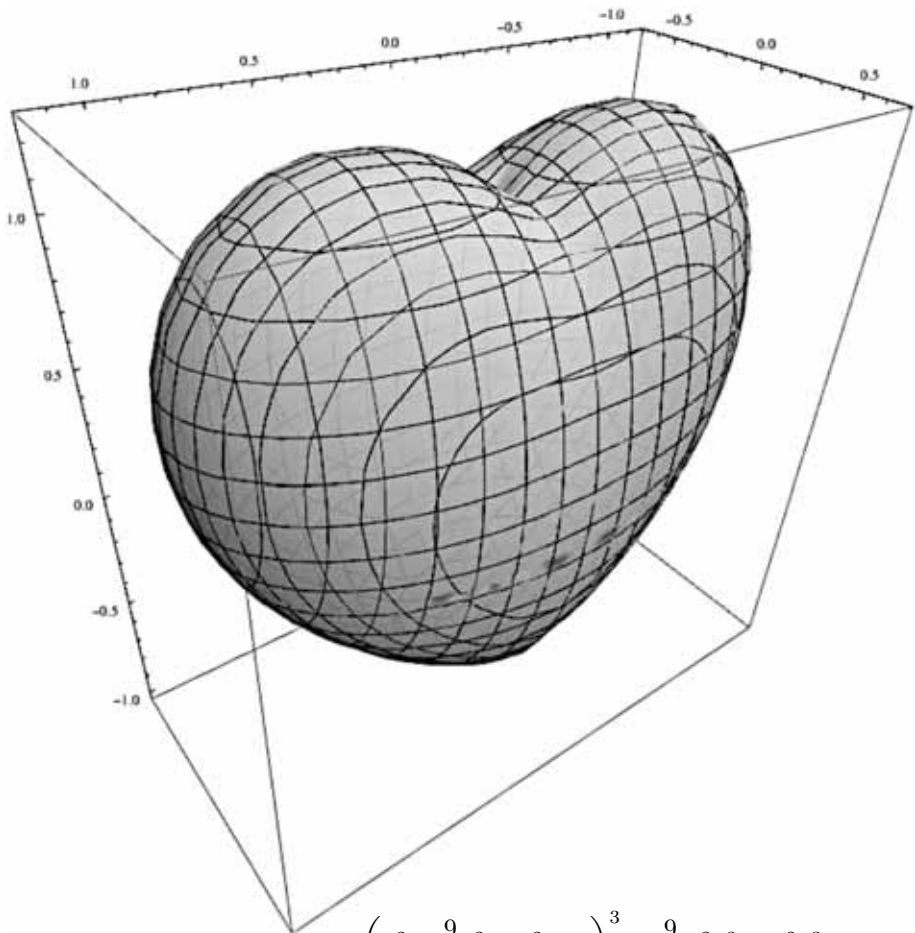
- M_{11} : quadruply sharply transitive group of permutations of 11 elements
has $11 \times 10 \times 9 \times 8 = 7920$ elements
- M_{12} : quintuply sharply transitive group of permutations of 12 elements
has $12 \times 11 \times 10 \times 9 \times 8 = 95\,040$ elements
- M_{23} : quadruply transitive group of permutations of 23 elements
has $23 \times 22 \times 21 \times 20 \times 19 = 10\,200\,960$ elements
- M_{24} : quintuply transitive group of permutations of 24 elements
has $24 \times 23 \times 22 \times 21 \times 20 \times 19 = 244\,823\,040$ elements

All Mathieu Groups (which arise as symmetries of so called *Steiner Systems*) are *Sporadic Simple Groups*, the largest of which is the *Monster Group* of size $\approx 8 \times 10^{53}$. As John Baez concluded in [1], the new ideas above are rather *too symmetric* to have an application in music theory. On the other hand there are still many possibilities to explore the music of mathematics...

References

This article was inspired by an Internet Blog by John Baez [1] and is based on research and examples described in the following articles:

- [1] Internet Blog by J. Baez, June 2006
<http://math.ucr.edu/home/baez/week234.html/>.
- [2] T. M. Fiore, *Music and Mathematics*,
<http://www.math.uchicago.edu/~fiore/1/musictotal.ps/>.
- [3] A. Crans, T. Fiore and R. Satyendra, *Musical Actions of Dihedral Groups*,
<http://www.crm.es/Publications/08/Pr822.pdf>.
- [4] T. Fiore and R. Satyendra, Generalized Contextual Groups,
http://mto.societymusictheory.org/issues/mto.05.11.3/mto.05.11.3.fiore_satyendra.pdf.



$$\left(x^2 + \frac{9}{4}y^2 + z^2 - 1\right)^3 - \frac{9}{80}y^2z^3 - x^2z^3 = 0$$

“Taubin’s Heart”

Love and Tensor Algebra

Stanisław Lem

translated by Michael Kandel

Stanisław Lem (1921–2006) was a Polish science fiction and philosophy writer. The following poem is an extract from *The Cyberiad* (1967), a series of humorous short stories from a mechanical universe inhabited by robots. It is what you are given when you ask *Electronic Bard* (an ultimate poem writing machine) to write a “love poem, lyrical, pastoral, and expressed in the language of pure mathematics”.

Come, let us hasten to a higher plane
Where dyads tread the fairy fields of Venn,
Their indices bedecked from one to n
Commingled in an endless Markov chain!

Come, every frustum longs to be a cone
And every vector dreams of matrices.
Hark to the gentle gradient of the breeze:
It whispers of a more ergodic zone.

In Riemann, Hilbert or in Banach space
Let superscripts and subscripts go their ways.
Our asymptotes no longer out of phase,
We shall encounter, counting, face to face.

I'll grant thee random access to my heart,
Thou'lt tell me all the constants of thy love;
And so we two shall all love's lemmas prove,
And in our bound partition never part.

For what did Cauchy know, or Christoffel,
Or Fourier, or any Boole or Euler,
Wielding their compasses, their pens and rulers,
Of thy supernal sinusoidal spell?

Cancel me not – for what then shall remain?
Abscissas some mantissas, modules, modes,
A root or two, a torus and a node:
The inverse of my verse, a null domain.

Ellipse of bliss, converge, O lips divine!
the product of four scalars it defines!
Cyberiad draws nigh, and the skew mind
Cuts capers like a happy haversine.

I see the eigenvalue in thine eye,
I hear the tender tensor in thy sigh.
Bernoulli would have been content to die,
Had he but known such $a^2 \cos 2\phi$!

The Editors are most grateful to Lem's heirs Barbara and Tomasz Lem and to Houghton Mifflin Harcourt Publishing for granting us permission to print this poem.

The Formula of Love

Sophie Dundovic, St John's College

The right time to get married

We seek to find the answers of many of life's most baffling questions by using mathematics. But are there some things for which we cannot devise a formula? Professor Tony Dooley from the University of New South Wales in Australia seems to think not. He has come up with a formula to calculate the optimal age at which a man should propose. But men beware, this method has only a 37% success rate. It may not be quite as accurate as we mathematicians would like, but given that the divorce rate is predicted to rise further this year, perhaps it is worth thinking about!

Whether we can turn matters of the heart into a few neat lines of equations can be fiercely debated. The method makes many assumptions, and clearly there is room for improvement, but this is an intriguing application to mathematics. Where n candidates of marriage material exist and become available in a random order it is optimal to maximise the probability of choosing the best candidate at the ideal time in life. Professor Dooley has done this by using a decision rule.

To make use of the **Fiancée Formula** a man should propose to the first woman he dates after he has reached his optimal proposal age, provided she is the best candidate so far. For those mathmos who do not want to leave marriage to fate the formula goes as follows:

Let n be the latest age at which you want to be married.
 Let p be the earliest age you would consider getting married.
 Your optimal proposal age is $p + 0.368(n - p)$.

The magic number 0.368 can be deduced using a method called *optimal stopping* and using the rather complicated sum

$$\sum_{r=2}^{n-k+1} \frac{(n-k)}{n} \frac{(n-k-1)}{n-1} \dots \frac{(n-k-r+2)}{n-r+2} \frac{k}{n-r+1} \frac{1}{r-1}$$

It is left to the reader to decide how deeply they feel mathematics can delve into

these unpredictable matters, but for those who are interested documented attempts to solve the ‘marriage problem’ go back to 1960 and there is further information on the University of New South Wales’ webpage.

How to find girls in the galaxy

However knowing the ideal age to get married still leaves one with the problem of finding a partner. Fortunately Peter Backus from the University of Warwick has developed a way to estimate the number of available and suitable partners using a rather unexpected version of the Drake equation.

Instead of estimating the number of highly evolved civilisations in our galaxy, he wants to estimate the number of women in London with whom he would likely find to be suitable partners. Let N be the population of the United Kingdom. We then have to estimate the following parameters which very much depend on how “picky you are”:

f_S = proportion of humans with the right **S**ex (male/female)

f_L = proportion of the above living in the right **L**ocation

f_A = proportion of the above at the right **A**ge

f_P = other **P**references such as education, physical attractiveness, etc.

Now $X = N \times f_S \times f_L \times f_A \times f_P$ gives the number of people in the population who meet your dating requirements. One can have an infinite number of additional preferences, but clearly if this is the case the probability of meeting a suitable mate will tend to zero. We have quantified the well known fact that the pickier you with partners the less likely you are to be ‘lucky in love’!

If you compare this to the size of the population as a whole, you see that the chance of meeting one of these suitors is slim. The probability that (given that suitor remains oblivious to the fact you have applied a formula to them) they will want to marry (or date) you reduces your chances of finding love even further. However, according to Backus’ publication (and general life experience) it is still more likely to find a perfect partner than to speak to an alien.

Writing about Mathematics

Dr Clifford A. Pickover

The Beauty and Utility of Mathematics

“An intelligent observer seeing mathematicians at work might conclude that they are devotees of exotic sects, pursuers of esoteric keys to the universe.”

— Philip Davis and Reuben Hersh, *The Mathematical Experience*

When I write mathematics books intended for popular audiences, I often mention how mathematics has permeated every field of scientific endeavour and plays an invaluable role in biology, physics, chemistry, economics, sociology, and engineering. Mathematics can be used to help explain the colours of a sunset or the architecture of our brains. Mathematics helps us build supersonic aircraft and roller coasters, simulate the flow of Earth’s natural resources, explore subatomic quantum realities, and image faraway galaxies. Mathematics has changed the way we look at the cosmos.

My personal desire to show how mathematics plays a role in a range of applications applies, in particular, to my most recent popular mathematics book, *The Math Book: From Pythagoras to the 57th Dimension* [1], which discusses and illustrates 250 milestones in the history of mathematics. In this book, I provide readers with a taste for mathematics using very few formulas, while stretching and exercising the imagination. However, the topics in this book are not mere curiosities with little value to the average reader. In fact, reports from the U.S. Department of Education suggest that successfully completing a mathematics class in high school results in better performance at university, *regardless of the degree* the student chooses to pursue [2].

The *usefulness* of mathematics allows us to build spaceships and investigate the geometry of our universe. Numbers may be our first means of communication with intelligent alien races. Some physicists, such as American theoretical physicist Michio Kaku, have even speculated that an understanding of higher dimensions and of topology — the study of shapes and their interrelationships — may someday allow us to escape our universe, when it ends in either great heat or cold, and then we could call all of space-time our home [3].

Simultaneous discoveries have often occurred in the history of mathematics. As I mention in my book *The Möbius Strip* [4], in 1858 the German mathematician August Möbius (1790–1868) simultaneously and independently discovered the Möbius strip (a wonderful twisted object with just one side) along with a contemporary scholar, the German mathematician Johann Benedict Listing (1808–1882). In a similar way, calculus was developed independently by English polymath Isaac Newton (1643–1727) and German mathematician Gottfried Wilhelm Leibniz (1646–1716). It is curious that so many discoveries in science were made at the same time by people working independently. For another example, British naturalists Charles Darwin (1809–1882) and Alfred Wallace (1823–1913) both developed the theory of evolution independently and simultaneously. Similarly, Hungarian mathematician János Bolyai (1802–1860) and Russian mathematician Nikolai Lobachevsky (1793–1856) seemed to have developed hyperbolic geometry independently, and at the same time.

Most likely, such simultaneous discoveries have occurred because the time was “ripe” for such discoveries, given humanity’s accumulated knowledge at the time the discoveries were made. Sometimes two scientists are stimulated by reading the same preliminary research of one of their contemporaries. On the other hand, mystics have suggested that a deeper meaning exists to such coincidences. Austrian biologist Paul Kammerer (1880–1926) wrote [5], “We thus arrive at the image of a world-mosaic or cosmic kaleidoscope, which, in spite of constant shufflings and rearrangements, also takes care of bringing like and like together.” He compared events in our world to the tops of ocean waves that seem isolated and unrelated. According to his controversial theory, we notice the tops of the waves, but beneath the surface some kind of synchronistic mechanism may exist that mysteriously connects events in our world and causes them to cluster.

Georges Ifrah in *The Universal History of Numbers* [6] discusses simultaneity when writing about the Mayan mathematics:

We therefore see yet again how people who have been widely separated in time or space have [...] been led to very similar if not identical results [...]. In some cases, the explanation for this may be found in contacts and influences between different groups of people [...]. The true explanation lies in what we have previously referred to as the profound unity of culture: the intelligence of homo sapiens is universal and its potential is remarkably uniform in all parts of the world.

Many entries in my books deal with whole numbers, or integers. The brilliant mathematician Paul Erdős (1913–1996) was fascinated by number theory — the study of integers — and he had no trouble posing problems, using integers, that were often simple to state but notoriously difficult to solve. Erdős believed that if one can

state a problem in mathematics that is unsolved for more than a century, then it is a problem in number theory.

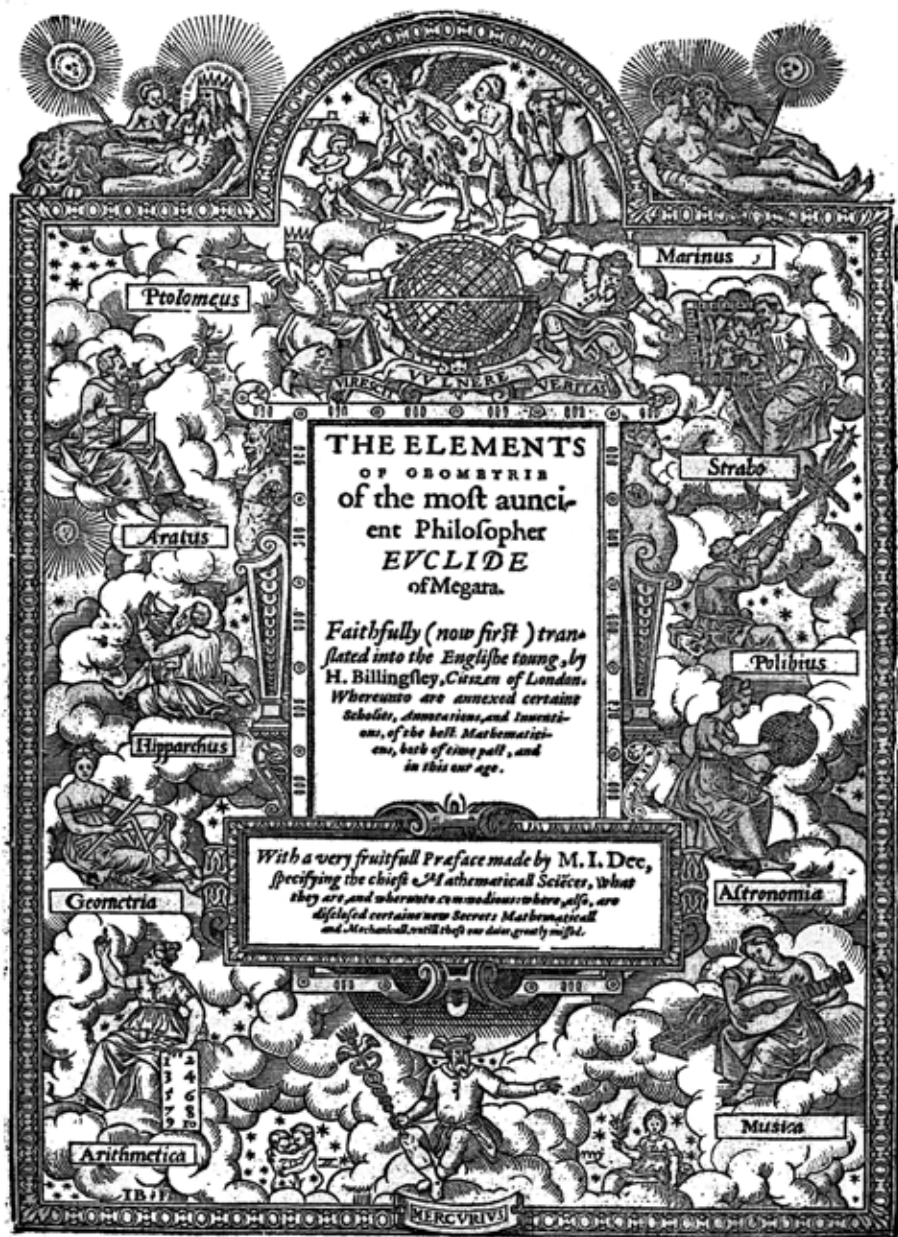
Ancient civilisations, such as the Greeks, had a deep fascination with numbers. Could it be that in difficult times numbers were the only constant thing in an ever-shifting world? To the Pythagoreans, an ancient Greek sect, numbers were tangible, immutable, comfortable, eternal — more reliable than friends, less threatening than Apollo and Zeus.

I enjoy pointing out to readers that many aspects of the universe can be expressed by whole numbers. Numerical patterns describe the arrangement of florets in a daisy, the reproduction of rabbits, the orbit of the planets, the harmonies of music, and the relationships between elements in the periodic table. Leopold Kronecker (1823–1891), a German algebraist and number theorist, once said, “The integers came from God and all else was man-made.” His implication was that the primary source of all mathematics is the integers.

Since the time of Pythagoras, the role of integer ratios in musical scales has been widely appreciated. More importantly, integers have been crucial in the evolution of humanity’s scientific understanding. For example, French chemist Antoine Lavoisier (1743–1794) discovered that chemical compounds are composed of fixed proportions of elements corresponding to the ratios of small integers. This was very strong evidence for the existence of atoms. In 1925, certain integer relations between the wavelengths of spectral lines emitted by excited atoms gave early clues to the structure of atoms. The near-integer ratios of atomic weights were evidence that the atomic nucleus is made up of an integer number of similar nucleons (protons and neutrons). The deviations from integer ratios led to the discovery of elemental isotopes (variants with nearly identical chemical behaviour but with different numbers of neutrons).

Small divergences in the atomic masses of pure isotopes from exact integers confirmed Einstein’s famous equation $E = mc^2$ and also the possibility of atomic bombs. Integers are everywhere in atomic physics. Integer relations are fundamental strands in the mathematical weave — or as German mathematician Carl Friedrich Gauss (1777–1855) said, “Mathematics is the queen of sciences — and number theory is the queen of mathematics.”

Our mathematical description of the universe grows forever, but our brains and language skills remain entrenched. New kinds of mathematics are being discovered or created all the time, but we need fresh ways to think and to understand. For example, in the last few years, mathematical proofs have been offered for famous problems in the history of mathematics, but the arguments have been far too long and complicated for experts to be certain they are correct. Mathematician Thomas Hales had to



Euclid's Elements

wait *five years* before expert reviewers of his geometry paper — submitted to the journal *Annals of Mathematics* — finally decided that they could find no errors and that the journal should publish Hale’s proof, but only with the disclaimer saying they were not certain it was right! Moreover, mathematicians like Keith Devlin have admitted in *The New York Times* [7] that “the story of mathematics has reached a stage of such abstraction that many of its frontier problems cannot even be understood by the experts.” If experts have such trouble, one can easily see the challenge of conveying this kind of information to a general audience. We do the best we can. Mathematicians can construct theories and perform computations, but they may not be sufficiently smart to fully comprehend, explain, or communicate these ideas.

A physics analogy is relevant here. When Werner Heisenberg worried that human beings might never truly understand atoms, Bohr was a bit more optimistic. He replied in the early 1920s, “I think we may yet be able to do so, but in the process we may have to learn what the word *understanding* really means.” Today, we use computers to help us reason beyond the limitations of our own intuition. In fact, experiments with computers are leading mathematicians to discoveries and insights never dreamed of before the ubiquity of computers. Computers and computer graphics allow mathematicians to discover results long before they can prove them formally, and open entirely new fields of mathematics. Even simple computer tools like spreadsheets give modern mathematicians power that Gauss, Euler, and Newton would have lusted after. As just one example, in the late 1990s, computer programs designed by David Bailey and Helaman Ferguson helped produce new formulas that related π to $\log 5$ and two other constants. As Erica Klarreich reports in *Science News* [8], once the computer had produced the formula, proving that it was correct was extremely easy. Often, simply knowing the answer is the largest hurdle to overcome when formulating a proof.

Mathematical theories have sometimes been used to predict phenomena that were not confirmed until years later. Maxwell’s Equations, for example, predicted radio waves. Einstein’s field equations suggested that gravity would bend light and that the universe is expanding. Physicist Paul Dirac once noted that the abstract mathematics we study now gives us a glimpse of physics in the future. In fact, his equations predicted the existence of antimatter, which was subsequently discovered. Similarly, mathematician Nikolai Lobachevsky said that “there is no branch of mathematics, however abstract, which may not someday be applied to the phenomena of the real world.”

In my books, readers often encounter various interesting geometries that have been thought to hold the keys to the universe. Galileo (1564–1642) suggested that “Nature’s great book is written in mathematical symbols.” Johannes Kepler (1571–1630) modeled the solar system with Platonic solids such as the dodecahedron. In the

1960s, physicist Eugene Wigner (1902–1995) was impressed with the “unreasonable effectiveness of mathematics in the natural sciences.” Large Lie groups, like E_8 may someday help us create a unified theory of physics. In 2007, Swedish-American cosmologist Max Tegmark published both scientific and popular articles on the mathematical universe hypothesis, which states that our physical reality is a mathematical structure — in other words, our universe is not just *described* by mathematics — it *is* mathematics.

Simplicity in Book Writing

“At every major step, physics has required, and frequently stimulated, the introduction of new mathematical tools and concepts. Our present understanding of the laws of physics, with their extreme precision and universality, is only possible in mathematical terms.”

— Sir Michael Atiyah, “Pulling the Strings,” *Nature*

One common characteristic of mathematicians is a passion for completeness — an urge to return to first principles to explain their works. As a result, readers must often wade through pages of background before getting to the essential findings. To avoid this problem, many popular mathematics writers, like me, find it useful to keep book entries short. Of course, this approach has some disadvantages. In just a few paragraphs, we cannot go into any depth on a subject. However, my philosophy is always to provide suggestions for further reading. While I sometimes list primary sources, I have often explicitly listed excellent secondary references that readers can often obtain more easily than older primary sources. Readers interested in pursuing any subject can use the references as a useful starting point.

When many individuals contribute to a mathematical idea, it can be a challenge to assign an appropriate historical date. Often, I have used the earliest reasonable date, but sometimes I have surveyed colleagues and decided to use the date when a concept gained particular prominence. For example, consider the Gray code, named after Frank Gray, a physicist at Bell Telephone Laboratories in the 1950s and 1960s. During this time, these kinds of codes gained particular prominence, partly due to his patent filed in 1947 and the rise of modern communications. Thus, I date the Gray Code entry to 1947, although it might also have been dated much earlier, because the roots of the idea go back to Émile Baudot (1845–1903), the French pioneer of the telegraph. In any case, I generally attempt to provide readers with a feel for the span of possible dates in each entry in a “Notes and Further Reading” section.

Scholars sometimes have disputes over attributing discoveries to individuals. For example, author Heinrich Dörrie cites four scholars who do not believe that a particu-

“Mathematics cultivates a perpetual state of wonder about the nature of mind, the limits of thoughts, and our place in this vast cosmos.”



lar version of Archimedes Cattle Problem is due to Archimedes, but he also cites four authors who believe that the problem *should* be attributed to him [9]! Scholars also dispute the authorship of Aristotle’s Wheel Paradox. Where possible, I mention such disputes either in the main text or the “Notes and Further Reading” section.

Even the naming of a theorem can be a tricky business. For example, mathematician Keith Devlin writes in his 2005 column for The Mathematical Association of America [10]:

Most mathematicians prove many theorems in their lives, and the process whereby their name gets attached to one of them is very haphazard. For instance, Euler, Gauss, and Fermat each proved hundreds of theorems, many of them important ones, and yet their names are attached to just a few of them. Sometimes theorems acquire names that are incorrect. Most famously, perhaps, Fermat almost certainly did not prove “Fermat’s Last theorem”; rather that name was attached by someone else, after his death, to a conjecture the French mathematician had scribbled in the margin of a textbook. And Pythagoras’s theorem was known long before Pythagoras came onto the scene.

In closing, let us note that mathematical discoveries provide a framework in which to explore the nature of reality, and mathematical tools allow scientists to make predictions about the universe; thus, the discoveries mentioned in books on the history of mathematics are among humanity's greatest achievements. For me, mathematics cultivates a perpetual state of wonder about the nature of mind, the limits of thoughts, and our place in this vast cosmos.

Our brains, which evolved to enable us to run from lions on the African savanna, may not be constructed to penetrate the infinite veil of reality. We may need mathematics, science, computers, brain augmentation, and even literature, art, and poetry to help us pierce the shrouds. For those of you who do embark on reading the *The Math Book* from cover to cover, look for the connections, gaze in awe at the evolution of ideas, and sail on the shoreless sea of imagination.

As the island of knowledge grows, the surface that makes contact with mystery expands. When major theories are overturned, what we thought was certain knowledge gives way, and knowledge touches upon mystery differently. This newly uncovered mystery may be humbling and unsettling, but it is the cost of truth. Creative scientists, philosophers, and poets thrive at this shoreline.

— W. Mark Richardson, "A Skeptic's Sense of Wonder," *Science*

References

- [1] C. Pickover, *The Math Book: From Pythagoras to the 57th Dimension, 250 Milestones in the History of Mathematics*, Sterling, New York, 2009.
- [2] K. Devlin, "Devlin's Angle: What Can Mathematics Do for the Businessperson?", The Mathematical Association of America, September 1999 (http://www.maa.org/devlin/devlin_9_99.html). Mentions the Department of Education findings that show the value of taking math courses in high school.
- [3] M. Kaku, *Hyperspace*, Anchor, New York, 1995.
- [4] C. Pickover, *The Möbius Strip*, Thunder's Mouth Press, New York, 2006.
- [5] P. Kammerer, *Das Gesetz der Serie (The Law of Seriality)*, Deutsche Verlags-Anstalt, Stuttgart, 1919.
- [6] G. Ifrah, *The Universal History of Numbers*, New Jersey: John Wiley & Sons, Hoboken, 1999.
- [7] S. Kruglinski, "When Even Mathematicians Don't Understand the Math", *The New York Times*, May 25, 2004 (<http://www.nytimes.com/2004/05/25/science/essay-when-even-mathematicians-don-t-understand-the-math.html>). Keith Devlin discusses how mathematics has become so complicated that it sometimes cannot be understood even by the experts.
- [8] E. Klarreich, "Math Lab: Computer Experiments are Transforming Mathematics", *Science News*, 165(17): 266-268, April 24, 2004.
- [9] H. Dörrie, *100 Great Problems of Elementary Mathematics: Their History and Solution*, Dover, New York, 1965.
- [10] K. Devlin, "Devlin's Angle: Naming Theorems", The Mathematical Association of America, September 2005 (http://www.maa.org/devlin/devlin_09_05.html).

Leaving the Textbook closed

Exciting Mathematics at School Level

James Gill and Tom Eaves, St John's College

What goes on in an undergraduate mathematics degree is not always well understood by laymen. Mathematics in schools is often taught by tackling many hundreds of short exercises on certain methods, and then reproducing them in the exam. This suggests that a maths degree will be more of the same, repetitive approach: a notion we are eager to dispel! With this in mind, we decided to visit Tom's secondary school during the Easter vacation, and carry out a couple of sessions on interesting maths with Years 10 and 11. We hope that this article will encourage you to consider trying something similar yourself.

Our aims for the lessons were first to get the children doing something exploratory with maths and, second, to give some inkling of the importance of proof, which is (perhaps rightly) neglected at GCSE.

The first, and more successful, session was based on a problem familiar to those who have grappled with IA probability examples sheets: Suppose Mary and Bob play a game in which they toss a fair coin until either the sequence HHH appears or the sequence THH appears. Who has a better chance of winning? One bright spark (irritatingly!) spotted the answer immediately, so we set him to deciding what winning sequence he would pick given a free choice. It is probably a good idea to have some kind of contingency task in mind if you do stumble across Terence Tao. The rest formed pairs and used the random number function on their calculators to simulate coins and play the game repeatedly. After fifteen or twenty minutes, we collated the results, observing – fortunately – a clear win for Bob in the classroom, although not in every pair (which lead to an interesting discussion about the nature of repeated experimentation). We then marshalled a class discussion to find a mathematical explanation for Bob's success.

This all seemed to go down very well, this kind of exploratory work being sadly rare in schools. The second half of the hour we filled with two rather counter-intuitive results: the famous Birthday problem, and the Newton-Pepys problem. Again, most

results in the classroom are thoroughly predictable, whereas an unexpected result is a key part of the joy of mathematics.

In the second session, we decided to show two proofs of Pythagoras' Theorem, which is commonly stated but not proven at GCSE. We then discussed what assumptions we had made in our proofs, and this led to a debate on why there are 180° in a triangle. When the proof of this was exhibited, we gave Euclid's axioms, and they seemed to enjoy this more thorough approach to explaining results. A couple of the students, quite rightly, showed some doubt at Euclid's parallel line axiom, which led to a brief discussion of spherical geometry and why it is important to fully understand assumptions you make. We then moved back to Pythagoras' Theorem, and asked them to confirm examples of Pythagorean Triples, before posing the question as to whether such triples exist for powers greater than 2. Obviously no-one could find one, but we provided our own examples for the students to test on their calculators, namely the incorrect result $1782^{12} + 1841^{12} = 1922^{12}$. We asked whether anyone could tell us why this was wrong, and after a minute or so we gave them a clue by suggesting they think about odd and even numbers. This helped to portray the importance of mathematical rigour, and why computer accuracy issues can very easily let you down. We concluded with a full statement of Fermat's Last Theorem, which astounded many of the students in the class.

At the end of each lesson, when we gave time for questions both about maths and a more general nature such as applications to Oxbridge (CUSU's excellent Target Schools resources proved extremely useful). The students felt they were benefiting greatly from access to a student able to answer these questions, and we were glad to see their enthusiasm showing at such an early stage of their education. Feedback from staff at the school was positive, the material seemed to have come across well. We thoroughly enjoyed the day; if you think you might too, then why not see if a local school might value your input? It is a rewarding experience not to be missed lightly!



STIMULUS is a community service programme which gives Cambridge University students the opportunity to work with pupils in local primary and secondary schools, helping with Maths, Science, ICT or Technology lessons.

You will have the opportunity either to help as assistant teacher in the classroom, or take responsibility for a small group of struggling or very able pupils.

If you are interested in joining, please visit our website <https://stimulus.maths.org/>.

A mathematical Interlude

6 Extracts from 60 Issues full of Humour and Fun

It is tradition for Eureka to include recreational problems and mathematical humour. In particular, authors liked to submit their research in poems rather than prose. Here is a selection of six of some of the most amusing articles from 60 issues of Eureka.

A Mathematical Crossword (E.P.H. and C.H.B.) Eureka 1, 1939

- Across

 - 1. Half a horse.
 - 3. 9 down in different order.
 - 5. 11 times 3 down.
 - 6. 2 down plus 12 down reversed.
 - 8. Square.
 - 10. Prime.
 - 11. 2-figure number.
 - 12. A regular solid.
 - 14. Lord Nelson.
 - 15. 1 across plus 13 down.
- Down

 - 1. Magazine without the printers.
 - 2. Perfection.
 - 3. One-eleventh of 5 across.
 - 4. Square.
 - 7. Without the grace of Noel Coward.
 - 9. The French half of the horse.
 - 12. Odd.
 - 13. Prime.

1	2	3	4	
5			6	7
8	9		10	
11		12	13	
14			15	

An Alphabet (“Pluto”) Eureka 10, 1948

A for **Analysis**, first the list
Of subjects whose purpose is usually missed.

B is for **Body**, an object most frigid
Which even in heat waves stays perfectly rigid.

C is for **Conic**: oh! common of curves,
It crops up so often it gets on your nerves.

D is for ∇^2 , for div and for det,
And several others we try to forget.

E for ϵ that’s greater than nought.
This magical symbol will save us much thought.

F is for **Field**; not where buttercups grow,
But where magnets and charges bring currents in tow.

G is for **Gravity**, dear to us all,
Or what else would happen to Newton’s old “ball”?

H is for **Hydromechanics**, a study
Of sources and streams — not the kind that are muddy!

I for **Infinity**, mythical place
Where circles and parallel lines show a face.

J for **Jacobian**, a pleasant device
For making the nastiest integral nice.

K is for **Kepler**, who left us some laws
Of planet'ry motion, effect but not cause.

L stands for so many things, that, in doubt,
I've chosen the **Limit** that's often about.

M is for **Matrix**, a mighty array —
If we didn't leave blanks we'd be writing all day.

N is for **Normal**, a misleading word,
For a “non-normal” normal's not even absurd!

O is for **Orbit**; we'll readily trace
The path of a body that's moving in space.

P is for **Particle** having no size;
It's wonderful what it can do when it tries.

Q is for **Quadric**, the Conic's big brother;
What's true for the one may be true for the other.

R is for **Rank**; but the Major is out,
For here it's the Minors we're worried about.

S is for **Sign** that's so often mislaid,
Explaining mistakes that should never be made.

T is for **Trip**.: how I wish that implied
A journey by car or a char-à-banc ride.

U for **Uniqueness**, important, I'm sure,
But the proofs of the theorems are rather too pure.

V is for **Vector**: all lecturers say
That the sum is the same if you take it each way.

W must obviously stand for a **Wave**;
The problem arises: “How does it behave?”

x, **y** and **z**, from their own point of view,
Are complaining; “We have far too much work to do;
It seems that for axes we're much better than
All the others; they use us whenever they can;
Though mathematicians may do as they like,
Beware! We may yet go on strike!”

A Contribution to the Mathematical Theory of Big Game Hunting (H. Pétard)

Eureka 13, 1953

Like many other branches of knowledge to which mathematical techniques have been applied in recent years, the Mathematical Theory of Big Game Hunting has a singularly happy unifying effect on the most diverse branches of the exact sciences. For the sake of simplicity of statement, we shall confine our attention to Lions (*Felis leo*) whose habitat is the Sahara Desert. The methods which we shall enumerate will easily be seen to be applicable, with obvious formal modifications, to other carnivores and to other portions of the globe.

1. The Method of Inverse Geometry. We place a *spherical* cage in the desert, enter it, and lock it. We perform an inversion with respect to the cage. The lion is then in the Interior of the cage and we are outside.

2. The Bolzano-Weierstrass Method. Bisect the desert by a line running N-S. The lion is either In the E portion or in the W portion; let us suppose him to be in the W portion. Bisect this portion by a line running E-W. The lion is either in the N portion or the the S portion, let us suppose him to be in the N portion. We continue this process indefinitely, constructing a sufficiently strong fence about the chosen portion at each step. The diameter of the chosen portions approaches zero, so that the Lion is ultimately surrounded by a fence of arbitrarily small perimeter.

3. The “*Mengentheoretisch*” Method. We observe that the desert is a separable space. It therefore contains an enumerable dense set of points, from which can be extracted a sequence having the lion as limit, We then approach the lion stealthily along this sequence, bearing with us suitable equipment.

4. The Peano Method. Construct, by standard methods, a continuous curve passing through every point of the desert. It has been shown that it is possible to traverse such a curve in an arbitrarily short time. Armed with a spear, we traverse the curve in a time shorter than that in which a lion can move his own length.

5. The Topological Method. We observe that a lion has at least the connectivity of the torus. We translate the desert into four-space. It is then possible to carry out such a deformation that the lion can be returned to three-space in a knotted condition. He is then helpless.

6. The Dirac Method. We observe that wild lions are, *ipso facto*, not observable in the Sahara Desert. Consequently, if there are any lions in the Sahara, they are tame. The capture of a tame lion may be left as an exercise for the reader.

7. The Schrödinger Method. At any given moment there is a positive probability that there is a lion in the cage. Sit down and wait.

8. The Thermodynamical Method. We construct a semipermeable membrane, permeable to everything except lions, and sweep it across the desert.

Chessmas (R. Schwarzenberger)

Eureka 22, 1959

Seven mathematicians were once shipwrecked on an island. They immediately set to work mining chalk, painting blackboards, weaving dusters and carving chessmen so that they would not have to abandon the mode of life which they found so congenial. Their housing problems were easily solved: around the cliffs of the island were seven caves each with access to a path along the shore, and easily reached from the Lone Pine which marked the centre of the island.

Each evening they would meet in one of the caves and decide where each should sleep that night by the following device: two by two they would play chess until a game was won or lost. Then the winner would leave (clockwise) for the neighbouring cave. The loser (anticlockwise) would do the same. As soon as more than one person arrived at a cave the procedure would be repeated. Every night, no matter in what order the games were played, or how long they took, fourteen games were played to a result before the seven mathematicians slept separate and undisturbed.

But one of the mathematicians (wiser than the rest) slept longer than the others. For each night as the chess started he would withdraw to a corner and let the other six choose their partners. Knowing that in a few hours everyone would have left the cave he went straight to sleep. Even when someone arrived eager to play chess, he slept on, knowing that soon a second would arrive and that they would play chess and both depart.

The younger mathematicians met by the Lone Pine to discuss this antisocial behaviour and to find ways of ensuring that all should play more chess. Surely the system could be modified so that more than a mere fourteen games were played each night. And this is what they decided: Instead of starting all at the same cave each of the seven mathematicians would go to the Lone Pine and roll a stone to decide at which cave he should begin. As soon as two arrived at a cave chess would begin and the winner and loser would move as before. By this method, it was hoped, more than fourteen games would sometimes be played. If by any chance no games were played a special holiday called "Chessmas" would be declared the next day.

In high hopes of Chessmas the stone rolling began. Alas! That night the chess continued until the dawn and would have gone on forever had they been faithful to their plan. If only, they said, we had been shipwrecked on a shore of infinite extent and unlimited equidistant caves. Then we would have variety but never more than twenty games each night.

The younger mathematicians were not put off. They constructed an eighth cave and worked the plan just as before. Chessmas now occurs several times a year and a special holiday of seven days has been promised should there be a recurrence of the disaster which caused the building of the eighth cave. So far this hasn't happened, nor indeed has a night with more than twenty games. Can either occur?

The reader is invited to check the statements made and to solve the questions raised. Will it do him any good? Yes, if ever he is shipwrecked with $2n$ other unfortunate mathematicians.

Ode to the Negative Gaussian Curvature of Potato Crisps (Colin Vout)

Eureka 36, 1973

Of all the unsolved problems that
 Confront us in this world,
 The biggest mystery to me
 Is: why are crisps so curled?

Their curvature is negative;
 Whichever way they went
 At first, you'll find to compensate
 They're oppositely bent.

Now, when it's plunged into the oil
 How does a crisp react?
 Does it expand, and buckle up,
 Or does the thing contract?

Or does It seek to minimise
 Its surface area?
 So, like a soap-film, there would be
 No max- or minima.

Considering an element:
 If forces balance out,
 Then ∇^2 crisp is zero and
 The same thing comes about.

But still, some crisps have, locally,
 A curving more than nought;
 Though by and large its sign will be
 A minus, as it ought.

Imagine, now, the heated vat;
 The oil begins to bubble—
 A sliced potato enters and
 Proceeds to bend up double.

Perhaps uneven heating makes
 One side shrink rather more;
 Then if the crisp should overturn
 It curls up, as before.

And so the curve is negative;
 But now we wonder what
 Would happen should it not reverse,
 For then the curve is not.

And anyway the temperature
 Is constant, I feel sure;
 For otherwise some overcook
 While other crisps stay raw.

Or do the bubbles, rising up,
 Distort the crisp that way?
 But crisps are sometimes bent in half;
 Explain that, if you may.

The facts we know about the world
 Should help us pass this hurdle;
 Or is this an example of
 The theorem proved by Gödel?

O Archimedes, answer this.
 Our patron and our hero:
**Why is the curvature of crisps
 So clearly less than zero?**

Footnote: A physicist has suggested a reason for the phenomenon in question is that the inner part of a potato contains more water than the outer; therefore it shrinks more on cooking; therefore there is in effect a circular frame holding the surface within it open; therefore the crisp is analogue to an open universe; and it is well known that a universe is open iff its Gaussian curvature is negative or zero. (How about a general cosmology of crisps?)

From the Mathematical Innovations
Catalogue (Chris Cummins)

Eureka 56, 2004

Klein Salad-Dressing Bottle

Keep oil on the inside, vinegar
on the outside.

Quantum Surfboard

Warning: Do not use on
unrestricted wavefunctions.

Random Walk Generator

Comprises stereotypical
mathmo, half-pint of
larger.

Anthropomorphiser

Ascribes human qualities and
emotions to functions, sets, num-
bers etc. Not for use on mathmos.

Dr Leader's Evil Adversary Self-Defense Kit

Sprays arbitrarily small epsilons
over a range of delta meters.

Book of Mispelt Adz for Pedents

Hours of fun for mathmos.

Escheristic Perpetual-Motion Machine

The ball rolling down an infinite slope
generates enough energy to power a
light bulb. New! Uphill version: Uses
two 1.5V AA batteries per day. The
ideal gift for someone you dislike.

Epsilon Magnifier

Sick of struggling with tiny epsilons? The revolu-
tionary new epsilon magnifier simplifies analy-
sis by increasing all epsilons to values > 1 .

COMING SOON:

3D Random Walk Generator

Also includes centrifuge, trampoline.
Accessories: reflecting barriers, ab-
sorbing barriers, Extra-absorbing bar-
riers for mopping up resulting spills.

Calculus-removing toothpaste

Guaranteed opaque.

**Matching Set of
Pathological Cases**

For the more experienced
traveller, save money with
our nowhere-dense set of
luggage.

The Millennium Prize Problems

A Series of Talks by the Archimedeans

Throughout the next year, the Archimedeans are hosting a series of lectures on the seven Millennium Prize Problems. For this occasion we would like to tell you a bit more about their history.

The Millennium Prize Problems are a selection of seven of the most important and most difficult unsolved problems in Mathematics. They were stated by the Clay Mathematics institute in 2000 and tie in with a list of 23 problems which were listed by David Hilbert in 1900. Five of these 23 problems are still unsolved and the Riemann Hypothesis has been included in both Hilbert's and the Millennium problems.

The Clay Institute will award \$ 1,000,000 for a correct solution.

The Riemann Hypothesis

One of the most famous unsolved problem in mathematics states that all nontrivial zeros of the analytical continuation of the Riemann zeta function have a real part of $1/2$. A proof (or counterexample) of the hypothesis will have significant implications regarding the distribution of prime numbers and other results in number theory.

P versus NP

Can a computer quickly (that is, in polynomial time) find a solution to a given problem (these problems are called P), given that it can verify a certain solution quickly (these problems are called NP)? In other words, is NP a subset of P? The P versus NP problem the most important open question in theoretical computer science and has far-reaching consequences many areas.

Yang–Mills existence and the Mass Gap

Yang–Mills theory is a generalisation of the Maxwell laws of electromagnetism. It has some solutions which travel at the speed of light, i.e. its quantum version should describe massless particles such as gluons. However, the postulated effect of “colour confinement” only permits bound states of gluons to form massive particles. This is called the mass gap.

The Hodge conjecture

The Hodge conjecture states that for projective algebraic varieties, Hodge cycles are rational linear combinations of algebraic cycles.

The Poincaré Conjecture (proven)

Every 2-dimensional surface which is both compact and simply connected is equivalent to a sphere. The Poincaré conjecture states that this is also true for spheres with 3-dimensional surfaces – a central problem in classifying 3-manifolds. The same question had long been solved for all dimensions above three.

The Poincaré conjecture was proved by Grigori Perelman in 2003, however he declined both the Millennium award and a Fields Medal.

Navier–Stokes Equations

The Navier–Stokes equations describe the motion of fluids and are not very well understood. To solve the problem, you have to develop a mathematical theory to give insight into these equations.

The Birch and Swinnerton-Dyer Conjecture

This conjecture deals with equation defining elliptic curves over the rational numbers. It states that there is a simple way to tell whether such equations have a finite or infinite number of rational solutions. Hilbert's 10th problem dealt with a more general type of equations, and it was proven that there is no way to decide whether a given equation even has any solutions.

You can find more details about our series of talks on page 8 – they promise to be most interesting and an amazing opportunity to hear some of the best mathematical speakers.

By the way: this page is **NOT** the annual Archimedean Problems Dive. However if you sent us a correct solution to any of the problems above (excluding the Poincaré conjecture) we would be delighted to publish it in the next issue of Eureka!

*Philipp Legner,
St John's College*

Careers for Mathematicians

Here's some good news. Unlike at many other Universities, you won't have to lose any time from studying Mathematics in order to learn careers related stuff as part of the curriculum here at Cambridge. This will undoubtedly help you to become a better mathematician, but unfortunately might leave you with some disadvantages in competing for opportunities after Cambridge.

What is the "careers related stuff" that intrudes into the curriculum at many other Universities? Much of it is based on a simple careers model. It goes like this. First understand yourself. Think about your skills, your knowledge, your personality, your interests, what motivates you, your preferred style of working, your short, medium and longer term goals in life, where you want to live and so on. Second, research the wide range of different options that could be open to you beyond your first degree – further education as well as employment or even time out. Then match what these require against the knowledge of yourself and select the paths that are right for you. Finally plan, and then implement, the necessary actions to move forward in your chosen area/s. A key part of the planning includes getting an understanding of how selection processes work for your chosen route, because, believe it or not, your Cambridge degree studies alone probably won't even get you in the door for an interview. You need to understand how to put together an application form or CV that works, and learn how to be convincing at interviews and assessment centres.

Here is some more good news. You can still do all this "careers related stuff" at Cambridge. We just provide it in a different way. There is a Careers Service here which offers a huge free programme of events, workshops, skill sessions, information, advice and guidance throughout the year, covering all of the above. The only difference is that it is there for you to dip into and to use as you wish outside the curriculum. Moreover, it is available to you as a current undergraduate or postgraduate student or at any time after you graduate.

So, if you haven't yet found or used the Careers Service (even though you are paying for us with your fees) then start here: www.careers.cam.ac.uk. Please also visit us at Stuart House, Mill Lane, Cambridge (next to Mill Lane Lecture Rooms) and take a look at our Library and the many free publications there, particularly the best free publication in Cambridge – our book on CVs and Cover Letters.



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Book Reviews

Sphere Packing, Lewis Carroll, and Reversi: Martin Gardner's New Mathematical Dimensions

Martin Gardner

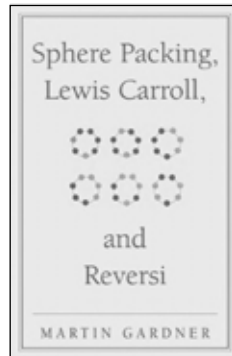
ISBN: 9780521747011, Price: £9.99 (paperback)

It is indisputable that Martin Gardner is one of the very best authors of recreational mathematics. For 25 years, he wrote the inspiring “Mathematical Games and Recreations” columns in *Scientific America*, as well as publishing more than 60 books, many of which were best sellers.

Sphere Packing, Lewis Carroll, and Reversi comprises a collection of 20 of the most interesting of these columns, ranging from board games and the four-colour theorem to interesting properties of π . This updated edition also includes solutions and addenda for the individual chapters.

I especially enjoyed chapter 9 about “Mathemagic”, where here describes a number of mathematical tricks that can be done by moving along playing cards or random closed curves, or by rotating soda crackers. The book contains many other ideas, problems and thoughts regarding geometry, logic, probability and other fields.

by Philipp Legner, St John's College

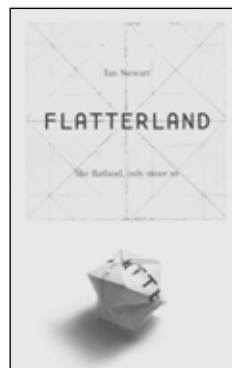


Flutterland: Like Flatland, Only More So

Ian Stewart

ISBN: 9780738204420, Price: £16.95 (paperback)

117 years is a long time to wait for a sequel, but *Flutterland* does its best to rekindle Edwin Abbott's two-dimensional satirical fire from *Flatland*, with Stewart employing a similarly witty style. No longer restricted to two dimensions, Stewart takes us on a geometric mindbender, visiting higher dimensions and fractional dimensions to illustrate fractals, topology, projective and hyperbolic geometry before exploring contemporary issues in physics such as time travel, quantum mechanics, black holes and relativity.



Our heroine student is Victoria (Vikki) Line, a teenage line, and our hero teacher the vastly knowledgeable “Space Hopper”, and through the immensely powerful Virtual Unreality device they traverse the Mathiverse, spurring the story along with a mixture of twee puns, edgy dialogue and cutesy anecdotes. The book’s level is that of accessible to anyone with hunger, desire and mathematical curiosity, entertaining, and informative enough for all but the most serious of readers. Despite the story petering out towards the end, with the physics consuming Vikki and the Hopper’s limelight, it’s nonetheless a learned, jocular and most of all enjoyable read.

by Conor Travers, St John’s College

Introduction to Continuum Mechanics

Sudhakar Nair

ISBN: 9780521875622, Price: £40.00 (hardback)

This text approaches continuum mechanics by developing the idea of describing an object’s position both before and after a deformation, and discussing the changes in the forces experienced through this motion. Through doing this it develops ideas from an early stage by focusing on basis transformations and their relation to area and volume changes. This seems to be a standard and appropriate approach to solid bodies, but makes the chapter on fluid mechanics somewhat cumbersome for students familiar with the approach of the IB fluids course.

Although the book is based on a graduate course, it opens with a few chapters that essentially recap our first year Vector Calculus course and then quickly moves on to more complicated and subtle applications. The focus in this book is to use suffix notation, which is a concise way of expressing mathematical reasoning, but the reader can’t help thinking that the significance of certain results would be more easily appreciated if they were stated in dyadic notation. Indeed, the motivations behind certain arguments were regularly absent. There are problem questions at the end of each chapter, which despite often being separated from the relevant learning material by some distance, are of an appropriately challenging standard. This text achieves exactly what would be expected from its title, providing a good introduction to a wide range of topics within this field, allowing the reader to quickly become fluent in this area’s language and techniques.

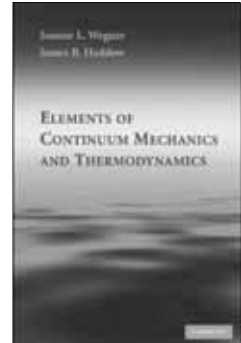
by Tom Eaves, St John’s College



Elements of Continuum Mechanics and Thermodynamics

Joanne L. Weger, James B. Haddow

ISBN: 9780521866323, Price: £40.00 (hardback)



This textbook covers similar material to that of Nair's Introduction to Continuum Mechanics, as discussed above. This text has a more motivational approach, with each section briefly discussing the physical significance of the method and of the results. The introduction again recaps Vector Calculus, but it also focuses heavily on basis transformations, providing a more detailed account as may be found in Linear Algebra. This text uses dyadic notation more often than the above text and, from an undergraduate perspective, gives more immediate visual understanding as to the meaning of its results. Again, the end of chapter exercises are interesting and of an appropriate volume, if slightly easier than in the previous text.

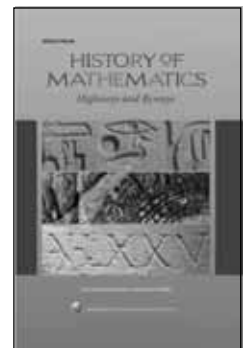
It does, however, have one oddity: its frequent references to Mathematica. Especially in the opening chapters, every other page contains a line similar to 'and [the eigenvalues] can be determined directly by using Mathematica' which at first sounds rather obtuse. Later in the book however, as is made clear in the blurb, Mathematica is used to emphasise examples and solutions and gives a refreshing if somewhat unusual feature. To summarise, the two textbooks cover similar material but in different styles. Both provide a good introduction to this field, and to choose between them is a matter of personal preference with regard to mathematical style, and of course mathematical background.

by Tom Eaves, St John's College

History of Mathematics: Highways and Byways

Amy Dahan-Dalmedico, Jeanne Peiffer, Sanford Segal

ISBN: 9780883855621, Price: £40.00 (hardback)



In lectures of the mathematical trips, topics are usually presented in a most logical order possible: starting with basic axioms and definitions and then deducing more complex properties and theorems. However this is very different from the way Mathematics was developed over time. This book gives an illuminating insight into how mathematical ideas evolved, and why they were developed at the time they were.

In 8 chapters, the authors cover topics such as Greek Mathematics, the development of Algebra, Analysis (in particular the notion of limits) and Geometry, leading to the crossover of all three subjects in complex numbers

and abstract linear algebra. The chapters include many info-boxes with definitions, theorems and short outlines of proofs.

Although the book contains little information about number theory, topology, probability and most of applied mathematics, I recommend it to anybody with an interest in the history of Mathematics.

by Philipp Legner, St John's College



The Ultimate Painting
by Drop Artists, 1966

The Archimedean Problems Drives Hints and Solutions

Question 1 (2009)

The answer is $A - B = -78.8\%$.

Question 2 (2009)

First observe that $f(x) = \sqrt{(x-4)^2 + (x^2-3)^2} - \sqrt{x^2 + (x^2-2)^2}$. In Cartesian coordinates, let $A = (4,3)$, $B = (0,2)$ and let $M = (x, x^2)$ be a point on the parabola $y = x^2$. Then clearly $S = MA - MB \leq AB = 2\sqrt{5}$.

Question 3 (2009)

- (a) These are the odd primes written in base-6. The next two are: 25, 31.
- (b) Each term in this sequence is the number of letters when the previous number is written out (in English). For example, there are six letters in 'twelve', so the next number is 6. There is some choice for the first number, but a possible solution is given: 73, 4.
- (c) The n th term of this sequence is $(n-1)^n$ for $n = 1, 2, \dots$. Therefore: 15625, 279936.
- (d) These are the numbers with an odd number of 1's in their binary expansion: 11, 13.
- (e) These are the numbers between twin primes: 42, 60.

Question 4 (2009)

Observe that for all natural numbers k , exactly two of $4k$, $4k+1$, $4k+2$, $4k+3$ are magic numbers, and their sum is $8k+3$ (This observation can be easily proved). Let $k = 1, 2, \dots, 1003$. This gives rise to 2006 magic numbers in the interval $[4, 4015]$. 1 and 2 are not magic numbers, but 3 is, since $3 = (11)_2$. Similarly we see that 4016 is not a magic number but 4017 is, since $4016 = (111110110000)_2$ and $4017 = (111110110001)_2$. Hence the sum of the first 2008 magic numbers is $8 \times (1 + 2 + \dots + 1003) + 3 \times 1003 + 3 + 4017 = 4035077$.

Question 7 (2009)

For $k \geq 2$ we have

$$\begin{aligned}
 2(\sqrt{k+1} - \sqrt{k}) &= \frac{2}{\sqrt{k+1} - \sqrt{k}} < \frac{1}{\sqrt{k}} < \frac{2}{\sqrt{k} + \sqrt{k-1}} = 2(\sqrt{k} - \sqrt{k-1}) \\
 \Rightarrow x &> 2 \sum_{k=2}^{1010025} (\sqrt{k+1} - \sqrt{k}) + 1 = 2(\sqrt{1005^2 + 1} - \sqrt{2}) + 1 \\
 &> 2 \times 2005 - 2\sqrt{2} + 1 = 2011 - 2\sqrt{2} > 2011 - 2 \times 1.5 = 2008
 \end{aligned}$$

We also have

$$x < 2 \sum_{k=2}^{1010025} (\sqrt{k} - \sqrt{k-1}) + 1 = 2(\sqrt{1005^2} - 1) + 1 = 2 \times 1004 + 1 = 2009$$

thus the integer part of x is 2008.

Question 6 (2009)

To solve this question we must express the expectation in a convenient way:

$$\begin{aligned}
 E[\text{number of apples}] &= \sum_{i=1}^6 E[\text{number of apples from column } i] \\
 &= \sum_{i=1}^6 E[\text{number of apples from column } i] P[\text{still have the apples at } \mathbf{B}] \\
 &= \sum_{i=1}^6 E[\text{number of apples from column } i] 2^{i-7}
 \end{aligned}$$

So we can redraw the orchard, where the number of apples in each tree is weighted by the probability that you still have them when you reach **B**:

	1/64	5/32	0	3/4	1/4	2	
	3/64	3/16	9/16	1/8	1/2	5/2	
	1/16	9/32	1/2	1/4	0	1	
A	5/64	1/32	1/4	3/8	2	5/2	B
	3/64	3/32	1/8	1/8	9/4	1	
	1/32	1/16	5/16	7/8	3/4	7/2	
	9/64	1/32	1/8	1	9/4	4	

Working backwards from **B**, we can show that the greatest we can achieve is $391/64$.

Question 8

Richard might pick any of the 9 balls, as might Ed (independently of Richard), so we have 81 elements in our sample space. When $y = 1, 2, 3, 4, 5$ any value of x in $\{1, 2, \dots, 9\}$ satisfies the inequality. For $y = 6$ we need $x = 3, 4, \dots, 9$ i.e. have 7 choices. For $y = 7$ we need $x = 5, 6, \dots, 9$ i.e. have 5 choices. For $y = 8$ we need $x = 7, 8, 9$ i.e. have 4 choices. Finally for $y = 9$ we need $x = 9$ i.e. have one choice only. Thus the probability is

$$\frac{5 \times 9 + 7 + 5 + 3 + 1}{81} = \frac{61}{81}.$$

Question 9

Carl Friedrich Gauss - Disquisitiones Arithmeticae

Isaac Newton - Arithmetica Universalis

Leonhard Euler - Vollständige Anleitung zur Algebra

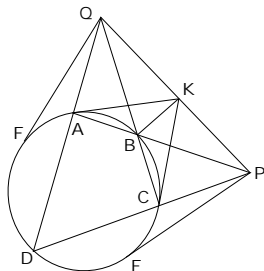
Daniel Bernoulli - Hydrodynamique

Gottfried Leibniz - Explication de l'Arithmétique Binaire

Question 10 (2009)

Take a point K on PQ such that $QA \times QD = QK \times QP$, so A, K, P, D are on a circle. Since A, B, C, D are on a circle we get $\sphericalangle QKA = \sphericalangle QDP = \sphericalangle QBA$, thus Q, K, B, A are on a circle. Therefore $\sphericalangle QPD = \sphericalangle QAK = \sphericalangle QBK$, thus B, K, P, C are on a circle. Therefore $\sphericalangle KCP = \sphericalangle KBP = \sphericalangle AQK$, so K, C, D, Q are on a circle. It follows that $PE^2 = PC \times PD = PK \times PQ$, so

$$\begin{aligned}
 PF^2 + PE^2 &= QK \times QP + KP \times QP = QP^2 \\
 \Rightarrow QP &= \sqrt{17^2 + 19^2} = \sqrt{650} = 5\sqrt{26}.
 \end{aligned}$$



Question 12

Of the eleven teams, eight picked 0 and hence each won one point, while three teams tried 1 and thus received nothing.

**Question 2 (2010)**

Dudeney, H. The Canterbury Puzzles. London: Thomas Nelson and Sons, Ltd. 1919

Volume considerations give at most 25 pieces. In reality, only 24 fit. Cut off half an inch from the longest side, then the block has dimensions $7\frac{1}{2}'' \times 4'' \times 3\frac{3}{4}''$. Cutting that into three slabs of $1\frac{1}{4}''$ produces three rectangles $7\frac{1}{2}'' \times 4''$, to be divided into rectangles of $2\frac{1}{2}'' \times 1\frac{1}{2}''$. It is easy to see that eight smaller pieces fit (note that $4 = 2\frac{1}{2} + 1\frac{1}{2}$ while $7\frac{1}{2} = 5 \times 1\frac{1}{2} = 3 \times 2\frac{1}{2}$).

Question 3 (2010)

Points to be awarded by the length of the maximal increasing subsequence.

- B 1209.6 = number of seconds in a millifortnight
- E 1727 = year of Isaac Newton's death
- G 1728 = great gross
- C 1729 = the smallest number expressible as the sum of two positive cubes in two different ways
- F 1836 = (rest mass of a proton) / (rest mass of an electron)
- D 1936 = year of the foundation of Archimedean
- A 2056 = magic constant of 16×16 normal magic square

Question 4

Gardner, M. More Mathematical Puzzles and Diversions. London: G. Bell and Sons Ltd, 1963

It is in Jones's best interest to wait until he has a single opponent, and then aim a first shot at him; until then he should fire in the air. The other opponents will first shoot at each other.

Smith's probability of survival if he fires first at Brown is $1/2$ (he will kill Brown, will survive Jones' attack with probability $1/2$, and will then kill Jones). If Brown fires first at Smith, the probability is reduced to $1/5 \times 1/2 = 1/10$. Since these two orders are equally likely, the overall survival probability is $1/4 + 1/20 = 3/10$.

Brown's chance of outliving Smith is $2/5$ (the probability that he goes first times the probability that he kills Smith). He now faces Jones, who fires first. Assuming that Brown survived n attacks from Jones already, his chance of surviving the $(n+1)$ st one is $1/2$ and his chance of killing Jones afterwards and terminating the process is $4/5$. Therefore, his chance of surviving Jones is $4/10 + (4/10)^2 + (4/10)^3 + \dots = 4/9$. His overall chance of surviving is then $2/5 \times 4/9 = 8/45$.

The probability that Jones is the last man standing is $1 - 3/10 - 8/45 = 47/90$.

Question 5 (2010)

USA Mathematics Talent Search, 1998-99

We rewrite the ellipse as

$$\frac{(x-20)^2}{20 \times 2010} + \frac{(y-10)^2}{10 \times 2010} = 1,$$

and draw the lines through its centre $(20,10)$. We consider the 5 areas labelled in the diagram as A, B, C, D, E . Due to the symmetry about the centre of the ellipse, the area we are looking for

is $2(A - B + D - E)$. Moreover, we see that $A = C + E$ and $D = B + C$, so the answer is in fact $4C$. Since C is bounded by the axes and the centre, the answer is $4 \times 20 \times 10 = 800$.

Question 6

Fujimura, K. The Tokyo Puzzles. London: Frederick Muller Limited, 1981

See to the right.

$$\begin{array}{r} \boxed{9} - \boxed{5} = \boxed{4} \\ \times \\ \boxed{6} / \boxed{3} = \boxed{2} \\ = \end{array}$$

Question 7

Fujimura, K. The Tokyo Puzzles. London: Frederick Muller Limited, 1981

6 moves are necessary:

$$\boxed{1} + \boxed{7} = \boxed{8}$$

$$(2,2)(1) \rightarrow (2,1)(2) \rightarrow (1,1)(4) \rightarrow (1,2)(1) \rightarrow (2,2)(2) \rightarrow (2,1)(4) \rightarrow (3,1)(6).$$

Question 10

USA Mathematics Talent Search, 1999-00

The pile is 36 rows high. Inside the pile, in a tetrahedral packing, each row is $\frac{2}{3}\sqrt{6}$. The total height of the pile is

$$(36 - 1) \times \frac{2}{3}\sqrt{6} + 2 = \left(\frac{70\sqrt{6}}{3} + 2 \right) \text{ cm.}$$

Question 8

- (1) 1,1,4,9,25,64,169,441,... (squares of the Fibonacci sequence)
- (2) 1,11,21,1211,111221,312211,13112221,... (self-describing sequence – read it out-loud)
- (3) 1,2,4,7,28,33,198,205,1640,... $(1 + 1, (1 + 1) * 2, (1 + 1) * 2 + 3, ((1 + 1) * 2 + 3) * 4, \dots)$
- (4) 3,3,5,4,4,3,5,5,... (number of letters in “one”, “two”, “three”, etc.)
- (5) 1,2,5,10,20,50,100,200 (current UK coins)
- (6) 1,11,4,4,20 (diagonal sequence of the above answers)

Question 11

1202	-	Leonardo Fibonacci	-	Book of the Abacus
1644	-	René Descartes	-	Principia philosophiae
1705	-	Edmond Halley	-	Synopsis Astronomiae Cometicae
1724	-	Abraham De Moivre	-	Annuities on Lives
1730	-	James Stirling	-	The Differential Method
1754	-	Sir Isaac Newton	-	An Historical Account of Two Notable Corruptions of Scripture
1801	-	Carl Friedrich Gauss	-	Disquisitiones Arithmeticae
1823	-	Augustin-Louis Cauchy	-	Le Calcul infinitésimal
1847	-	George Boole	-	The Mathematical Analysis of Logic

Question 12

What do *you* think?

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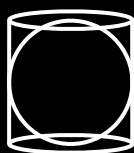
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